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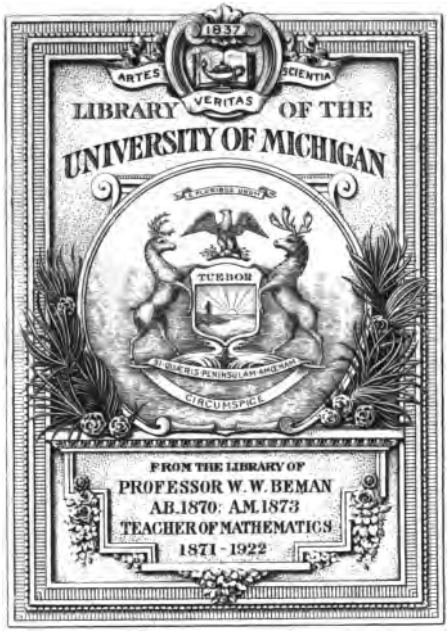
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2632. (The Editor.)—Prove that

$$1 \cdot 2 \cdot 3 \ldots n \equiv n! < 2^{\frac{1}{2}n(n-1)}. \quad \ldots\ldots\ldots 51$$

2665. (Professor Crofton, F.R.S.) — 1. Given any closed convex boundary, without salient points: if dS be any element of the external area; T, T′ the tangents drawn from dS to the boundary; θ the angle they contain; and if the integration extend over the whole external area; then prove that

$$\iint \frac{\sin\theta}{\mathrm{TT}'}\, d\mathrm{S} = 2\pi^2.$$

2. If the integration extend over the annular space between the proposed boundary and an outer line along which θ has a constant value (α), the value of the integral is $2\pi(\pi - \alpha)$.

3. If the given boundary has salient points, then for every such point where the bounding line changes direction abruptly through an angle A, we must subtract $\frac{1}{2}\mathrm{A}^2$ from the above values: then the value is $2\pi^2 - \frac{1}{2}\Sigma\mathrm{A}$. For instance, for a regular polygon of n sides, the value will be

$$2\pi^2(1 - 1/n). \quad \ldots\ldots 58$$

2874. (The late T. Cotterill, M.A.)—Seven points on a cubic locus have an opposite point on the curve: *i.e.*, a variable cubic through seven given points cuts a fixed cubic through the same points, in two other points collinear with a point on the fixed cubic. Construct for the opposite point when the fixed cubic breaks up into a conic through five points and a line through two. 48

3481. (Professor Cayley, F.R.S.)—Find, in the Hamiltonian form t

$$\frac{d\eta}{dt} = \frac{\mathrm{dH}}{d\varpi}, \quad \frac{d\varpi}{dt} = -\frac{d\mathrm{H}}{d\eta}, \text{ \&c.,}$$

the equations for the motion of a particle acted on by a central force. 57

3557 and 3615. (J. B. Sanders.)—Find the velocity and angle of elevation of a ball in order that it may be 100 feet above the ground at the distance of one mile. If the ball is projected from the top of a tower 200 feet high, at an angle of elevation of 60°, with a velocity of 50 feet, find the range on the horizontal plane passing through the foot of the tower, and the time of flight. 37

3768. (J. B. Sanders.) — A conical vessel of base-radius r, and height h, is filled with fluid; find the time in which the fluid will descend through half the height, if an orifice be made at the vertex of the cone, when the axis is vertical..... 123

3865. (Artemas Martin, LL.D.) — The boundary lines of a quadrangular county ABCD are AB = a, BC = b, CD = c, DA = d, and the diagonal AC = e; find the most equitable position for the county seat, supposing the county uniformly populated. 85

3973. (Professor Hudson, M.A.)—A flat circular disc of radius a is projected upon a rough horizontal table, which is such that the friction upon an element α is $cV^2m\alpha$, where V is the velocity of the element, m the mass of a unit of area: find the path of the centre of the disc.

If the initial velocity of the centre of gravity and the angular velocity of the disc be u_0, ω_0, prove that the velocity u and angular velocity ω, at any subsequent time, satisfy the equation $\left(\frac{3u^2-a^2\omega^2}{3u_0^2-a^2\omega_0^2}\right)^2 = \frac{u^2\omega}{u_0^2\omega_0}$.

..... 35

4014. (C. H. Hinton, M.A.)—Let ABC be a triangle, and through a on BC draw a straight line cutting AC in b, AB in c. From centre b distance bC, draw a circle; from centre c distance cB, draw a circle cutting the other in P, P′. Bisect the external or internal angle of the triangle Pbc or P′bc by a straight line cutting ab in x. Let x' be a point on another straight line $ab'c'$ similarly obtained, and so on; then x, x', x''... lie on a straight line passing through A. 76

4112. (C. H. Hinton, M.A.) — Given two points, one on each side of a triangle, the distances of each from the base-vertex of that side, the angle at which the line joining these points cuts the base, and the ratio of the sides: construct the triangle. 41

4114. (A. Renshaw.)—If OX be a line drawn outside a circle whose centre is R, and RH the perpendicular from R on OX; also, if P_1Q, P_2S, P_3T be drawn at right angles to OX from any three points P_1, P_2, P_3 on the circumference; prove that

$2RH\,(OS.P_1W - OQ.P_2Z - OT.P_1F) = OP_2^2.QT - OP_1^2.ST - OP_3^2.QS$,

where P_1W, P_2Z, P_1F are the differences of the perpendiculars from P_1P_3, P_2P_3, P_1P_2, and QT, ST, QS the segments of OX between the feet of perpendiculars on OX. 67

4132. (Matthew Collins, LL.D.)—The equation of a curve of the nth order being P + Q + R + S + &c. = 0, where $P = ax^{n-2}y^2 + a'x^{n-3}y^3 + \&c.$, $Q = bx^{n-1} + b'x^{n-2}y + \&c.$, $R = cx^{n-2} + c'x^{n-3}y + \&c.$; show that Euler has fallen into error in giving $ay^2 + bx = 0$ as the equation of the asymptotic parabola, and find the *correct* equation. 50

4160. (C. H. Hinton, M.A.)—If there be two circles cut by a straight line, and the points where each circle is cut be joined to its centre and the joining line produced,—(1) show that the four triangles thus formed on portions of the straight line as bases have their sides all in the same ratio; (2) how must the straight line move that the triangles thus formed have their sides in the same ratio for every position? 28

4193. (Professor Evans, M.A.)—At the station A, the apparent angular elevation of a meteor B, whose distance from the earth's surface is one-nth of the earth's radius, is θ. Supposing the earth to be a perfect sphere, find the exact distance from A to B. 42

4219. (The Editor.)—Given the radii of three circles inscribed in a triangle, so that each circle touches the other two, and two sides of the triangle; find the sides of the triangle. 77

6810. (J. R. Harris, M.A.)—A chain is lying in a given curve on a smooth table, and an impulse is communicated to it along the tangent at one end. Apply the theorem that the kinetic energy of the resulting motion is a minimum to determine the impulsive tension at any point of the chain. If the energy of the resulting motion be distributed uniformly along the chain, and be λ per unit of length, prove that the tension and the mass per unit of length are given by

$$T = \sqrt{(\tfrac{1}{2}\lambda\mu\rho)}\,[\sqrt{(\rho+2s)}-\sqrt{(\rho-2s)}]\,;\ \mu\rho \propto \exp.\int \frac{1}{\rho}\,d\,(\rho^2-4s^2)^{\frac{1}{2}}\,;$$

ρ being the radius of curvature of the chain at a point whose distance from the free end is s. [exp. $x \equiv e^x$.] 113

7099. (The Editor.)—Sum (1) the series

$$\frac{\sin^3\theta}{1^4} - \frac{\sin^3 3\theta}{3^4} + \frac{\sin^3 5\theta}{5^4} - \&c.$$

between the limits $\theta = 0$, $\theta = 2\pi$, and $\theta = \frac{1}{8}\pi$, $\theta = \frac{3}{8}\pi$; and hence show (2) that $\pi^4 = \dfrac{3\,.\,2^{\frac{1}{2}}}{11}\left(\dfrac{1}{1^4}-\dfrac{1}{3^4}+\dfrac{1}{5^4}-\dfrac{1}{7^4}+\&c.\right)$. 92

7251. (Professor Wolstenholme, M.A., Sc.D.)—At each point P of an epicycloid (or hypocycloid) is drawn the equiangular spiral of closest contact (four-pointic), S being its pole; prove that (1) S, P are inverse points with respect to the fixed circle of the epicycloid; (2) the arc traced out by S during one complete revolution of the moving circle is

$$2a\left(1-\frac{a^2}{(a+2b)^2}\right)^{\frac{1}{2}}\log\left(\frac{(a+b)^{\frac{1}{2}}+b^{\frac{1}{2}}}{(a+b)^{\frac{1}{2}}-b^{\frac{1}{2}}}\right) \text{ for the epicycloid,}$$

and $4a\left(\dfrac{a^2}{(a-2b)^2}-1\right)^{\frac{1}{2}}\sin^{-1}\left(\dfrac{b}{a}\right)^{\frac{1}{2}}$ or $4a\left(\dfrac{a^2}{(2b-a)^2}-1\right)\cos^{-1}\left(\dfrac{b}{a}\right)$

or the hypocycloid, according as $2b <$ or $> a$; a, b being the radii of the fixed and moving circles respectively; (3) the radii of curvature at S, P of the curves are as OS : SP − OS; and (4) there will be inflexions on the locus of S whenever $OS\sqrt{2} = a$; whence by inversion it appears that, in any epicycloid, when the central distance is $a\sqrt{2}$, the circle of curvature will pass through the centre of the fixed circle..... 68

7887. (Professor Mukhopâdhyây, M.A.)—Express, in finite form,

$$S \equiv \frac{x}{1^2} + \frac{x^3}{3^2} + \frac{x^5}{5^2} + \ldots \frac{x^{2n-1}}{(2n-1)^2} + \ldots \textit{ad inf.}$$ 26

8669. (The Editor.)—If a point P be taken on a rectangular hyperbola, and, QQ′ being any diameter, circles be drawn touching the hyperbola at P and passing through Q, Q′ respectively; prove that these circles will be equal..... 115

9030. (Professor Ignacio Beyens. Suggérée par la Quest. 8326.)—Si du centre de gravité on abaisse perpendiculaires GL, GL′, GL″ sur les hauteurs d'un triangle ABC, et si de l'orthocentre O on abaisse des perpendiculaires sur les médianes, OM, OM′, OM″; les deux triangles LL′L″, MM′M″ sont *homologiques* et le centre d'*homologie* est le point K de Lemoine du triangle ABC..... 121

9031. (Professor Pramathanath Dather, M.A.)—Trace the curve $xy^2+x^3+ay^2=0$, and find its first negative pedal with respect to the origin. 27

9088. (A. R. Johnson, M.A.)—Express (1), in terms of the fundamental covariants and invariants of a system of three conics, the covariant which, equated to zero, gives the locus of the point the tangents from which to the three conics form a system in involution; also, find (2) the equation to the scroll generated by the lines touching each of three quadrics; and (3) express, in terms of the fundamental invariants of three quadrics, the invariant whose vanishing is the condition that the three quadrics should have a common self-conjugate pentahedron..... 64

9148. (Professor Asutosh Mukhopadhyay.) — Prove that (1) the reciprocal polar of the evolute of the hypocycloid

$$\left(\frac{x}{a}\right)^{\frac{2}{3}}+\left(\frac{y}{b}\right)^{\frac{2}{3}}=1 \quad \text{is} \quad \left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)^3=\left(\frac{x^4}{a^2}-\frac{y^4}{b^2}\right)^2,$$

the radius of inversion being unity; and (2), if $a=b$, the polar equation of the reciprocal polar is $r=a\sec 2\theta$. 112

9162. (Professor Byomakesa Chakravarti, M.A.) — Show that the reciprocal polar of the evolute of the hyperbola $b^2x^2-a^2y^2=a^2b^2$, with regard to the circle described on the line joining the foci as diameter, is the curve $(a^2/x^2)-(b^2/y^2)=1$, which, when the hyperbola is equilateral, becomes $r^2\sin 2\theta+4a^2\cot 2\theta=0$. 40

9204. (Professor Byomakesa Chakravarti, M.A.) — Show that, if k be the radius of inversion, the reciprocal polar of the evolute of

$$(x/a)^m+(y/b)^m=1 \quad \text{is} \quad k^2\left\{(+x/a)^{m/(m-1)}+(-y/b)^{m/(m-1)}\right\}^{1/m}$$
$$=(b/y)(+x/a)^{1/(m-1)}+(ax)(-y/b)^{1/(m-1)}. \text{ ..... } 124$$

9306. (Professor Ignacio Beyens.)—Si dans un triangle quelconque ABC, AA′ est la médiane correspondante au côté BC, et l'on mène à AA′ une parallèle qui coupe BC en H, AC en N, BA en D; démontrer que les points symétriques de H par rapport aux milieux de NC, BD sont deux points H_1, H_2 symétriques par rapport au sommet A du triangle proposé. 127

9406. (W. J. Barton, M.A.)—Prove that, if $R=2r$, the triangle is equilateral. 46

9447. (S. Tebay, B.A.)—Draw a circle to trisect the circumference of a given circle and cut it at right angles. 62

9891. (R. W. D. Christie.)—Draw a straight line cutting two concentric circles, so that the part intercepted by them is divided into three equal portions. 109

9916. (R. Knowles, B.A.)—A rectangular hyperbola touches a given parabola at P, and cuts it in one other point Q; prove that (1) the locus of its centre is a circle which (*a*) has its centre on the directrix; (*b*) has its diameter equal to the radius of curvature at P; (*c*) intersects the circle of curvature at P on the tangent at P; (2) the locus of the mid-point of PQ is a parabola, which touches the given parabola at P. 118

9999. (R. H. W. Whapham, B.A.)—Two ports B and C are in the same parallel of latitude (C being west of B); their common latitude being l, and their difference of longitude 2λ. Two ships set out to sail from B to C, one due West and the other along the great circle through B and C; if they sail uniformly, and both arrive at C at the same time, prove that their distance apart when they have gone half-way is

$$r\left\{\tfrac{1}{2}\pi - l - \tan^{-1}(\cot l \cos\lambda)\right\},$$

where r is the radius of the earth, and l expressed in circular measure. 63

10002. (Artemas Martin, LL.D.)—A speaks the truth x times out of y. We do not know the values of either x or y, but x may be 0, 1, 2, 3, 4, 5, ... y; and y may be 1, 2, 3, 4, ... n. A asserts the occurrence of an event about which we know nothing; find the probability of the truth of A's statement. 34

10023. (Professor Brasseur.)—Démontrer que tout nombre composé de 3^n chiffres 1 est un multiple de 3^n. 117

10068. (Professor de Longchamps.)—Résoudre l'équation

$$(x+b+c)(x+c+a)(x+a+b) - abcx/(a+b+c) = 0. \ldots\ldots 120$$

10080. (Professor Syamadas Mukhopâdyhây, B.A. Extension of Quest. 9849.)—Similar triangles are similarly drawn on the sides of a polygon as bases, all externally or all internally. If equal particles be placed at their vertices, the C.G. of the system will coincide with the C.G. of equal particles placed at the angular points of the polygon. 102

10082. (D. Biddle.)—The circumference of a circle being represented as to length, by a given straight line; show how (by *pure* geometry) to find the radius, with at least sufficient accuracy for any practical purpose. 76

10088 and 10122. (F. R. J. Hervey.)—(10088.) Prove that, if, for each of the four triangles formed by four lines, a line be drawn bisecting perpendicularly the distance from circumcentre to orthocentre, the four bisecting lines are concurrent.

(10122.) Supposing the nine-point circle of a triangle to vary so that its radius retains a constant ratio to the distances of its centre from circumcentre and orthocentre, show that it will always meet the lines of the triangle at the vertices of two triangles similar to the given and pedal triangles respectively. 37

10093. (R. H. W. Whapham, B.A.).—From any point P on the evolute of a parabola are drawn normals PQ, PQ′ to the parabola, Q being on the same side of the axis as P; prove that the distance from the vertex of the evolute to the foot of the perpendicular from P on the axis

is three times the part of the axis intercepted between this latter point and the normal PQ, and that it is also three times the part of the axis intercepted between the vertex of the evolute and the normal PQ..... 63

10095. (Rev. J. J. Milne, M.A.)—If G is the centroid of any triangle, prove that (1)

$\cot GAB + \cot GBC + \cot GCA = 3\cot\omega = \cot ABG + \cot BCG + \cot CAG$,

$\cot AGB + \cot BGC + \cot GCA + \cot\omega = 0$, $\cot\omega = \cot A + \cot B + \cot C$;

(2) if K be the symmedian point, that is, a point within the triangle such that $\angle KAC = \angle GAB$, and two similar relations, then

$\cot AKB + \cot BKC + \cot CKA + \frac{1}{2}\cot\omega + \frac{3}{2}\tan\omega = 0$..... 33

10106. (Professor Syamadas Mukhopâdhyây, B.A.)—P, Q, R are three points on a parabola, PR being parallel to the tangent at Q; r is the radius of a circle touching the tangents at P, Q, R, and r' is the radius of a corresponding circle touching the normals at P, Q, R. Prove (1) that $\tan\delta = r'/3r$, where δ is the angle between the normal and diameter at Q; and hence (2) deduce for any curve TRANSON'S formula $\tan\delta = \rho'/3\rho$, where δ is the angle of aberrancy at a point (A) of the curve, and ρ, ρ' the radii of curvature of the curve at the point (A), and of the evolute at the point corresponding to (A). 72

10124. (D. Biddle.)—A cubical die, of side a, lies at the bottom of a cylindrical dice-box, of radius r. The dots or pips being symmetrically arranged on their several sides and each of radius ρ, find the probability that one of those on the uppermost side of the cube is placed vertically over the centre of the box. 27

10135. (R. W. D. Christie.)—Prove that

$\sqrt{n} = 2^{\frac{1}{2}(n-1)} \cos a \cos 3a \cos 5a \ldots \cos\frac{1}{2}(n-1)\,.\,a$,

when n is odd, and $a = \pi/2n$. 120

10165. (W. J. Greenstreet, M.A.)—Find the locus of the vertices of the triangles circumscribing a fixed circle, and of which the orthocentre is a fixed point. 125

10174. (F. Bouvier.)—Dans un hexagone inscrit, les six circonférences passant par le centre du cercle circonscrit et par les extrémités de chaque côté se coupent deux à deux en trois points situés sur une circonférence passant par le centre du cercle circonscrit..... 121

10176. (B. F. Finkel, B.Sc.)—Inscribe a regular 17-gon in a circle. 28

10200. (Capitaine De Rocquigny.)—Neuf nombres entiers consécutifs sont disposés en carré de la manière suivante:

$n-4,\ n+2,\ n-2$ $n-1,\ n,\ n+4$ $n+3,\ n-3,\ n+1$	Le centre est occupé par le nombre du milieu; les autres nombres se succèdent, d'après la loi du cavalier, dans l'ordre de grandeur

croissante. Choisir n de manière que la valeur de ce *déterminant* du cavalier soit un carré parfait. 82

10210. (E. M. Langley, M.A.)—Find a point E within the triangle

ABC such that $l \cdot AE + m \cdot BE + n \cdot CE$ may be a minimum; l, m, n being any three constants such that two are together greater than the third. 43

10211. (E. Lemoine.)—Par les milieux A′, B′, C′ des côtés d'un triangle quelconque ABC, on mène les symétriques de BC, CA, AB par rapport à une direction donnée. Ces trois droites concourent en un point du cercle des neuf points du triangle ABC. 30

10229. (Professor Stammer.)—La perpendiculaire élevée au milieu du côté BC d'un triangle rencontre les deux autres côtés AB, AC en des points B′ et C′ tels que les segments BB′ et CC′ sont vus de tout point de la circonférence ABC sous des angles égaux ou supplémentaires. Autrement dit, si B′ et C′ sont deux points divisant un diamètre d'une circonférence en parties harmoniques, et si BC est une corde quelconque perpendiculaire à ce diamètre, les droites BB′, CC′ sont vues de tout point de la circonférence sous des angles égaux ou supplémentaires. 73

10240. (R. W. D. Christie.)—Prove that any power of any prime number equals either the sum or the difference of two squares. 115

10249. (Professor Lemoine.) — Construire un tétraèdre équifacial SABC, connaissant la droite sur laquelle est l'arête SA, le plan mené par SA parallèlement à BC, un point de BC, enfin un triangle LMN semblable à ABC. 50

10254. (L. W. Robinson, M.A.)—If $(1+x)^n = 1 + c_1x + c_2x^2 + \ldots$, prove that, according as n is even or odd,

$$\Delta \equiv \begin{vmatrix} 1, & c_1, & c_2 \ldots & c_n \\ c_1, & c_2, & c_3 \ldots & \\ \ldots & & \ldots & \ldots \\ c_n, & c_{n-1} & \ldots & \end{vmatrix} = 0 \quad \text{or} \quad 2^n.$$

..... 54

10256. (R. H. W. Whapham, B.A.)—From any point P on a semicubical parabola there are drawn normals PQ, PQ′ to the parabola; PN is drawn perpendicular to the axis, meeting it in N, and V is the vertex of the semicubical parabola: if PQ, PQ′ meet the axis in G, G′, respectively, show that $VN = 3NG = 3VG'$. 82

10264. (Professor Décamps.) — Si, dans un tétraèdre OABC, trirectangle en O, α représente l'angle dièdre formé par les faces ABC et OBC, on a $\tan^2\alpha = \tan^2 ABO + \tan^2 ACO$. 79

10276. (Rev. T. R. Terry, M.A., F.R.A.S.)—If $gh = V^2 \cot^2\alpha \log\sec\alpha$, where V, g, and α are constants, and $p = 2gxV^{-2}\tan^2\alpha$, find the value of the integrals

$$I = \frac{\sin\alpha}{V}\int_0^h \frac{dx}{\sqrt{e^{-p} - \cos^2\alpha}} \text{ and } I' = \frac{\sin\alpha}{V\cos^2\alpha}\int_0^h \frac{dx}{\sqrt{\sec^2\alpha - e^p}}.$$

..... 34

10288. (W. J. Greenstreet, M.A.)—Find the envelope of the sides of a triangle which is inscribed in an ellipse, and whose orthocentre is a fixed point. 83

10293. (Professor Russo.)—En un point quelconque de la base BC d'un triangle isoscèle ABC, on élève une perpendiculaire, qui rencontre AC en D, AB en E. Démontrer que le lieu du point de rencontre des droites BD, CE est une ellipse. 79

10298. (Professor Wolstenholme, M.A., Sc.D.) — An equilateral triangle is drawn circumscribing a given parabola $y^2 = 4ax$; prove that the envelope of its circumscribed circle is a hyperbola $3(x+a)^2 - y^2 = 12a^2$ similar to the hyperbola which is the locus of the angular points of the triangle. Of course the envelope of the inscribed circle is the similar hyperbola $3(x+a)^2 - y^2 = 3a^2$. [The locus of the angular points is well known to be the hyperbola $(3x+5a)^2 - 3y^2 = 16a^2$.] 55

10331. (D. Biddle.)—A rectangle of given dimensions is divided by a random straight line. Find the average ratio which the smaller portion bears to the larger. 57

10351. (Professor Syamadas Mukhopâdhyây, B.A.) — An infinite number of lines are drawn in a plane at random except in so far that they make with a fixed straight line an angle which only varies from α to $\alpha + \psi$, ψ being less than π. Show (1) that they intersect within any closed figure of area S at $N^2(\psi - \sin\psi)$ S points, where N is the number of these lines intersecting a convex figure of unit perimeter when they are not restricted in direction, that is, when $\psi = \pi$; and hence (2) deduce a simple proof of Professor CROFTON's celebrated theorem. 103

10355. (Colonel Thiry.)—Dans un triangle ABC, si AT est la tangente au cercle circonscrit, et si une sécante quelconque coupe AT, AB, AC en des points T′, B′, C′, démontrer que $AB : AC' - AC : AB' = BC : AT'$. 103

10357. (E. M. Langley, M.A.)—If the centre of a circle be the centroid of four points on its circumference, prove that (1) they are vertices of a rectangle; and hence (2) if

$$\sin\alpha + \sin\beta + \sin\gamma + \sin\delta = 0 = \cos\alpha + \cos\beta + \cos\gamma + \cos\delta,$$

then $\sin(2n+1)\alpha + \sin(2n+1)\beta + \sin(2n+1)\gamma + \sin(2n+1)\delta = 0$,

and $\cos(2n+1)\alpha + \cos(2n+1)\beta + \cos(2n+1)\gamma + \cos(2n+1)\delta = 0$;

also (3) enunciate and prove a corresponding theorem for three angles. 54

10369. (R. H. W. Whapham, B.A.) — Show that the last term of the equation whose roots are the squares of the differences of the roots of $x^n = 1$ is n^n 84

10371. (R. Knowles, B.A.)—The sides AB, CD of a quadrilateral inscribed in a conic, centre O, are parallel; the diagonals AC, BD intersect in G; a second conic, centre O′, is drawn through A, B, C, D; prove that the points O, O′, G are collinear. 80

10375. (Capitaine de Rocquigny.)—Quel que soit n, démontrer que l'un des deux nombres $3^{3n} + 2^{3n}$, $3^{3n} - 2^{3n}$ est divisible par 35. 121

10415. (Professor Steiner.)—Soient M, M′, les extrémités de deux diamètres conjugués OM, OM′ d'une ellipse. Démontrer que la somme $\tan^2 \frac{1}{2}FMF' + \tan^2 \frac{1}{2}FM'F'$ est constante 70

10416. (Professor G. B. Mathews.)—Prove that (1) the conditions that $10x^3-8g_2x-5g_3=0$, $315x^4-231g_2x^2+90g_3x+25g_2^2=0$, may be satisfied by the same value of x, is

$t^2-2\,.\,5^3\,.\,79t-5^3\,.\,11^3=0$, where $t=27g_2^3/(g_2^3-27g_3^2)$;

and (2) that $t=5^3\,.\,79\pm2^2\,.\,5\,.\,13\,.\,17\sqrt{5}=\pm5\sqrt{5}\,(4\mp\sqrt{5})^3\,(\sqrt{5}\pm2)^4$, where the upper or lower sign is to be taken throughout; and if we put $t=4\,(1-y^2)^3/y^4$, one value of y is $\frac{1}{16}\,(\sqrt{5}-1)^3$. 71

10424. (W. J. Barton, M.A.)—Show that, if the sum of two integers is a square, and the sum of their squares a square, (1) the sum of their cubes is the sum of two squares; (2) their product is always divisible by 12; (3) if they are both even, their product is divisible by 192..... 38

10436. (Professor Delahaye.)—Mener par les sommets d'un triangle trois droites concourantes déterminant sur les côtés trois segments non consécutifs égaux entre aux. 31

10438. (Colonel Servais). — Si en un point quelconque A d'une conique, on mène un cercle tangent dont le rayon soit double du rayon de courbure en ce point: démontrer que (1) le pôle P, par rapport au cercle, de la corde d'intersection BC de la conique et du cercle est sur la conique; (2) les deux droites BC et PA sont conjuguées harmoniques par rapport à la conique. 88

10447. (Professor de Longchamps.)—Soient une circonférence Δ et deux diamètres rectangulaires Δ', Δ''. D'un point M, de Δ, on abaisse des perpendiculaires MP′, MP″ sur ces diamètres: soient P le pôle de la droite P′P″ par rapport à Δ; Q, R, ses projections sur Δ', Δ''. Démontrer que QR est tangente à Δ. 25

10455. (Professor Déprez.) — Les milieux des côtés d'un triangle ABC, le centre I du cercle inscrit, le centre O du cercle circonscrit, et le centre de gravité du périmètre de ABC sont six points d'une même hyperbole équilatère..... 127

10456. (The Editor.)—On the base BC of a triangle find a point P such that AP^2 may have to BP . PC a given ratio. 117

10464. (Professor Zerr, M.A.)—Solve the equations

$x+y^2+z^3=21$, $x^2+y^3+z=45$, $x^3+y+z^2=71$. 49

10466. (R. H. W. Whapham, B.A.)—The two tangents from $x'y'$ to $x^2/a^2+y^2/b^2=1$ form with $lx+my+n=0$ a triangle circumscribing an ellipse; show that, if ξ, η is the middle point of $lx+my+n=0$, we have

$\xi : nx'-la^2=\eta : ny'-mb^2=-1 : lx'+my'-n$. 32

10469. (Professor Lemoine.)—Soient I, I_a, I_b, I_c les centres des cercles tangents aux trois côtés du triangle ABC, et soient T, T_a, T_b, T_c les points de concours des droites joignant A, B, C aux points de contact des côtés opposés avec l'un de ces cercles. Démontrer que les droites IT, I_aT_a, I_bT_b, I_cT_c concourent en un même point T, et trouver les coordonnées de ce point. 53

10471. (R. Knowles, B.A.)—In any triangle ABC, prove that (1) the tangent to the circumcircle at A, the symmedian from a, and the sides

AB, AC from a harmonic pencil; (2) the tangent, the side BC, and the perpendicular from the circumcentre on the symmedian are concurrent. 85

10472. (Prof. Bénézech.) — Dans un triangle donné ABC, mener, parallèlement à une direction donnée, une sécante B′C′ telle que BB′ = CC′, B′ et C′ étant les points de rencontre de cette sécante avec AB et AC. 72

10473. (Colonel Servais.)—Si m est un nombre entier, on a

$$\frac{1.3.5\ldots(2m+1)}{2.4.6\ldots(2m+2)} = \frac{1}{2} - m\frac{1}{2.4} + \frac{m(m-1)}{1.2}\cdot\frac{1.3}{2.4.6} - \ldots \quad \ldots(1),$$

$$\frac{2.4.6\ldots 2m}{3.5.7\ldots(2m+1)} = 1 - m\frac{1}{3} + \frac{m(m-1)}{1.2}\cdot\frac{1}{5} - \ldots \quad \ldots\ldots\ldots(2).$$

..... 35

10482. (Prof. Zerr.)—Give a formula for differentiating $y = x^{x^{x^{\&c.}}}$ and show that the same formula will answer for $y = a^{x^{x^{\&c.}}}$ and $y = e^{x^{x^{\&c.}}}$. 38

10484. (Professor Syéchnikoff.)—Résoudre les équations

$$x+y+z = a, \quad x^2+y^2+z^2 = a^2+2b^2, \quad x^3+y^3+z^3 = a^3.$$

..... 56

10485. (Professor Décamps.)—Résoudre les équations

$$x^2+y^2 = a, \quad z^2+u^2 = b, \quad xy+zu = c, \quad xu+yz = d. \ldots\ldots\ldots 39$$

10491. (Professor Mathews.)—Prove that, (1) if $Pa+Qb+Rc+Sd=0$, then the result of changing x, y, x', y' into $by-ay'$, $cx-ax'$, $dy-cy'$, $dx-bx'$, in the expression $f \equiv Pxy+Qxy'+Rx'y+Sx'y'$, is $(bc-ad)f$; and (2) that a similar theorem is true when

$$P : Q : R : S = d : -c : -b : a. \ldots\ldots\ldots 39$$

10492. (Professor Neuberg.) — Cinq forces parallèles se faisant équilibre, on transporte chacune d'elles au centre de la sphère passant par les quatre autres points d'application. Démontrer que l'équilibre existe encore, pourvu que les forces restent parallèles entre elles..... 30

10494. (Professor Wolstenholme, Sc.D.)—Prove that (1) the Lemoine-point of a maximum triangle inscribed in the ellipse $a^2y^2+b^2x^2 = a^2b^2$ is given by any two of the three equations

$$\frac{x}{a}\cos\alpha + \frac{y}{b}\sin\alpha + \frac{a^2-b^2}{2(a^2+b^2)}\cos 2\alpha = 0, \quad \frac{x}{a}\cos\beta + \&c., \quad \frac{x}{a}\cos\gamma + \&c.,$$

where α, β, γ are the excentric angles of the corners of the triangle; (2) at the Lemoine-point, $x/a\cos 3\alpha = y/b\sin 3\alpha = (b^2-a^2)/2(a^2+b^2)$; and that its locus is the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{(a^2-b^2)^2}{4(a^2+b^2)^2}$; and (3) that the Brocard-points are given by the two systems

$$\frac{x/a\cos\alpha + y/b\sin\alpha + \frac{1}{2}}{(a^2\sin^2\beta + b^2\cos^2\beta)} = \frac{x/a\cos\beta + y/b\sin\beta + \frac{1}{2}}{(a^2\sin^2\gamma + b^2\cos^2\gamma)} = \frac{x/a\cos\gamma + y/b\sin\gamma + \frac{1}{2}}{(a^2\sin^2\alpha + b^2\cos^2\alpha)}$$

$\frac{x/a\cos\alpha+y/b\sin\alpha-\frac{1}{2}}{(a^2\sin^2\gamma+b^2\cos^2\gamma)} = \frac{x/a\cos\beta+y/b\sin\beta-\frac{1}{2}}{(a^2\sin^2\alpha+b^2\cos^2\alpha)} = \frac{x/a\cos\gamma+y/b\sin\gamma-\frac{1}{2}}{(a^2\sin^2\beta+b^2\cos^2\beta)}$,
where $\cos\alpha+\cos\beta+\cos\gamma=0$, $\sin\alpha+\sin\beta+\sin\gamma=0$..... 111

10495. (Professor Morley.)—Let ABC be a triangle, and let F be a focus of the maximum inscribed ellipse. Denoting these points by the complex variables z_1, z_2, z_3, z, prove that (1) $\Sigma 1/(z-z_r)=0$, $(r=1,2,3)$; and hence (2) if AF meets the circle FBC again at H, FH = 3AF. ... 61

10505. (R. Knowles, B.A. Suggested by Quest. 10385.)—A, B, C, D, taken in order, are the angular points of a parallelogram; the circle ABC cuts AD in A′, and DC in C′; prove that A′D : A′C′ = A′C : A′B. 33

10506. (Professor Wolstenholme, Sc.D.)—Corresponding to each point P of a given ellipse (or hyperbola) whose foci are S, S′, is described another ellipse (or hyperbola) having a focus at S′, a vertex at P, and its major axis equal to the major axis of the given ellipse; prove that the only real common point of this conic and the given one are P, Q, where PQ is the chord normal at P. 52

10507. (J. White.)—A circle of given radius a rolls upon a fixed straight line, and from a given point in it tangents are drawn to this circle; prove that the loci of the centres of the four circles which can be described to touch the rolling circle, the given line, and the tangents from the given point, are two equal parabolas, and the curve $x^2(y-a)^2=4ay^3$. 78

10514. (Professor Genese, M.A. Suggested by Question 10497.)—A, B are fixed points of intersection of two conics; X, Y, the points of contact of a common tangent; and XY meets AB at T. Prove that the expression $XY\,(TA\,.\,TB)^{\frac{1}{2}}/(TX\,.\,TY\sin\theta)$, where θ is the angle between XY and AB, is independent of the choice of tangent. 52

10517. (Professor Steiner.)—Par un point fixe P l'axe focal d'une ellipse ou d'un hyperbole, on mène une sécante quelconque rencontrant la courbe en M. M′. Démontrer qu'il existe une constante r, telle que le produit $(FM+r)\,(FM'+r)$ a une valeur constante, F étant l'un des foyers. 80

10522. (Professor Neuberg.)—Soient un triangle équilatéral ABC et une circonférence concentrique; tous les triangles dont les sommets sont les projections, sur les côtés de ABC, d'un point mobile sur cette circonférence, ont même angle de Brocard. 51

10543. (R. Tucker, M.A.)—Eliminate t from the equations
$(2t^4-t^2+1)x+(2t^5-2t^3+t)y=4at^4$, $(t^4-t^2+2)tx-(t^4-2t^2+2)y=4at$.
..... 31

10556. (Professor Neuberg.)—Quatre forces parallèles, dont les points d'application sont situés dans un même plan, se font équilibre. On applique chacune d'elles au centre du cercle passant par les points d'application des trois autres. La direction des quatre forces restant la même, démontrer que l'équilibre existe également dans la nouvelle position. 25

10557. (Professor Nilkantha Sarkar.) — An electric cable is laid between two ports, and it is found that the gross annual receipts amount to 20 per cent. on the original outlay: the annual cost of repairs is 2 per cent. on the original cost, and the expense of working is found to be £500 a year in addition to 20 per cent. on the net receipts. After a year, a second cable is laid down, and it is found that the gross receipts and the expense of working are doubled, while the cost of repairs for the new cable is only 1 per cent., and that the *net* receipts now amount to £14,500. Find the cost of a cable and the net receipts for the first year. 36

10559. (Professor Lampe, LL.D.)—Let A be the upper end, F the foot, of a vertical staff $AF = l$, standing on the horizontal plane at F under the latitude ϕ; then, if the declination of the sun is supposed to be constant during a day, the shadow of A is known to describe in our latitudes one branch of an hyperbola the declination being $+\delta$, the other branch if the declination is $-\delta$. The asymptotes of all these hyperbolas for different days envelope a parabola tangent at its vertex to the straight line described by the shadow of A on the days of equinox, and having the latus rectum $4ltg\phi$. The locus of the extremities of the conjugate axes of all hyperbolas is again a parabola with the latus rectum $ltg\phi$. [These theorems hold, similarly, for the sections of a series of coaxial and concentric cones of revolution by an arbitrary plane.] 74

10560. (Professor Genese, M.A.)—If $\{a_1 b_2 c_3 \ldots\}$ be a determinant, and $a_\alpha b_\beta c_\gamma \ldots$ a term, prove the following rule for the sign of a term:—Let p be the number of suffixes after α and *less than* α; q, r, &c., the corresponding numbers for β, γ, &c.; then the sign of the term is $(-1)^{p+q+r+\ldots}$. 81

10561. (Professor Zerr.) — If F = (force) $= \mu r^{-5}(5r^2 + 8c^2)$, and a particle be projected from an apse at a distance c with the velocity from infinity, show that the equation of the orbit is $r = c\cos\frac{1}{2}\theta$. 56

10562. (Professor Morley.)—Let there be three parallel rectilinear vortices in an infinite mass of fluid. Let a cross section meet the vortices at ABC. Prove that (1) the motion of any vortex A is at right angles to AK, where K is the symmedian point; and (2) the particles at foci of the maximum ellipse inscribed in ABC are instantaneously at rest. 42

10563. (Professor Catalan.)—Résoudre, complètement, l'équation

$$(x^3 - 3qx + p^3 - 3pq)^2 - 4(px + q)^3 = 0. \ldots\ldots 49$$

10568. (Professor de Longchamps.)—Soient A, B, C trois points en ligne droite. Par A, on fait passer une droite mobile Δ, et des points B, C on abaisse, sur Δ, des perpendiculaires BB′, CC′. Trouver le lieu décrit par le point de concours des diagonales du trapèze ainsi formé. 40, 72

10569. (Professor Tarry.) — Le rapport anharmonique de quatre points A, B, C, D d'une conique est égal à la racine carrée du rapport anharmonique du faisceau P(ABCD) qui a pour centre le pôle P de la droite AB. 71

10570. (Professor Morize.)—Construire un triangle sachant que les côtés passent par trois points donnés en ligne droite;—que deux des sommets se trouvent sur deux parallèles données, et qu'enfin la médiane issue du troisième sommet a une longueur connue. 66

10572. (R. Tucker, M.A.)—Tangents to a parabola at the points P, Q, R, form a tangent-triangle *pqr*, whose sides make angles θ_1, θ_2, θ_3 with the axis of the curve. Show that (1) the rectangular coordinates of the "S"-point of *pqr* are given by

$$X = \frac{2a}{D}\left\{\sin(\theta_1+\theta_2+\theta_3)+\Sigma\sin(\theta_1+\theta_2-\theta_3) - \frac{\Sigma\sin(\theta_1+\theta_2-\theta_3)+\Sigma\sin(\theta_1+\theta_2-3\theta_3)}{2D'}\right.,$$

$$Y = \frac{a}{D}\left\{\frac{\Sigma\cos(\theta_1+\theta_2-\theta_3)-\Sigma\cos(\theta_1+\theta_2-3\theta_3)}{D'} - 4\cos(\theta_1+\theta_2+\theta_3)\right\},$$

where $D \equiv 8\sin\theta_1\sin\theta_2\sin\theta_3$, $D' \equiv 1-\cos(\theta_1-\theta_2)\cos(\theta_2-\theta_3)\cos(\theta_3-\theta_1)$; (2) if two of the tangents are drawn at the extremities of the L.R., then the locus of the "S"-points is

$$Y^2(2X+3a)+(2X+a)(X+a)^2=0;$$

and (3) if the angles made by the tangents are θ, $\frac{1}{2}\pi+\theta$, and $\frac{1}{2}\pi-2\theta$, then the locus is $4\lambda(4-\lambda)y^2 = a^2[\lambda^2-\lambda+4]^2$, where $2X+3a = a\lambda$..... 86

10576. (D. Edwardes, B.A.)—The boundaries of a membrane are two confocal ellipses, and are fixed in space. The density at any point varies inversely as the product of the distances of the point from the foci. Show that there are modes of vibration in which the nodal lines are ellipses confocal with the boundaries, the periods being given by

$$P = \frac{1}{i\pi}\left(\frac{2\pi M}{T}\log\frac{a+b}{a'+b'}\right)^{\frac{1}{2}},$$

where i is an integer, M the mass of the membrane, T the tension, a, b and a', b' are the semi-axes of the outer and inner boundaries. 100

10577. (H. J. Woodall.) — A certain complex quantity has the operation $(\ldots)^2$ repeated on it a number of times (producing the series, say, k, k^2, k^4, k^8, ... k^{2^n}, &c.). If the values recur, in a period of q, required an investigation of the form of the complex. 116

10578. (W. J. Greenstreet, M.A.)—A circle passes through a fixed point on an ellipse, and has its centre on the same curve; prove that (1) the circle envelopes a quartic; (2) the quartic has a cusp at the fixed point; and (3) the tangent at the cusp is a normal to the ellipse. 55

10579. (Morgan Brierley.)—A variable circle touches another given in magnitude and position, and also a line given in position; determine the magnitude and position of the variable circle when the line joining the centres makes a given angle with the given line. 46

10584. (E. M. Langley, M.A.)—O is any point in the plane of a triangle ABC. If AO, BO, CO cut the circumcircle again in A', B', C', and D, E, F are the projections of O on BC, CA, AB, show (1) that the triangles A'B'C', DEF are directly similar, and that O in A'B'C' corresponds with the isogonally inverse point of O in DEF; also, (2) use the theorem to demonstrate the existence of cosymmedian triangles having the sides of each proportional to the medians of the other. 128

10588. (Professor Neuberg.)—La droite AB est fixe; la droite A'B se meut parallélement à elle-même entre deux droites fixes OX, OY. Trouver le lieu du centre de similitude de deux figures semblables construites sur AB, A'B'. 45

10590. (Professor Genese, M.A.)—A, B are two points of a circle, O any point in its plane; OPQ is any secant, and AP, BQ meet at R. The locus of R is a conic through A, B. Prove that, by properly choosing the circle and point O, the locus may be made to coincide with any given conic through A, B. 41

10591. (Professor Zerr.)—Prove that the area of the triplicate-ratio circle is equal to the areas of the cosine and the Brocard circles. 101

10592. (Professor Morley.) — Rays of light parallel to an axis are reflected at acycloid. Show that the caustic is a cycloid. 94

10593. (Professor Catalan.) — Si, à un quadrilatère inscriptible Q on inscrit une infinité de quadrilatères P dont le périmètre soit minimum, ce minimum commun est une quatrième proportionnelle au rayon et aux diagonales de Q. 91

10594. (Professor Wolstenholme, M.A., Sc.D.) — On the parabola $y^2 = 4ax$ is taken the point P $(am^2, 2am)$, and the coordinates of a point Q are $[a(8+m^2), -\frac{2}{3}am]$; prove that the circle described with centre Q and radius QP will meet the parabola again in three points which are corners of an equilateral triangle, and that QP is always normal to the parabola $y^2 = 36a(x+12a)$. 48

10595. (Professor de Longchamps.)—On donne a et a_1; puis, on calcule $a_2, a_3, \ldots a_n$, par voie récurrente, au moyen de la formule

$$a(n-1)a_n = a_1a_{n-1} + a_2a_{n-2} + \ldots + a_{n-1}a_1;$$

démontrer que les nombres $a_1, a_2, \ldots a_n$ forment une progression géométrique..... 88

10600. (Professor Hudson, M.A.)—If the mid-centre of a triangle lies on a side of the triangle, prove that the difference between the angles to which that side is adjacent is a right angle. [The mid-centre is the centre of the mid-circle, that is to say, of the circle that passes through the middle points of the sides.] 78

10602. (Professor Coupeau.)—Si deux cordes AB, CD d'un cercle sont telles que les droites qui joignent A et B au milieu de CD soient symétriques par rapport à CD, les droites joignant C et D au milieu de AB sont également symétriques par rapport à AB. 96

10605. (The Editor.)—Through four given points draw (1) a conic such that one of its chords of intersection with a given conic may pass through a fifth given point, and show (2) what the problem becomes when two of the four points are the circular points at infinity. 93

10606. (R. Tucker, M.A.) — P $(am_1^2, 2am_1)$, Q (m_2), R (m_3) are three co-normal points on a parabola: show that the rectangular coordinates of

the "S"-point of PQR are given by $\Sigma(m)^2 . Y = -6am_1m_2m_3$,
$2\Sigma(m^2) . [\Sigma(m^2) + 24] X = a[\{\Sigma(m^2)\}^2 . \{\Sigma(m^2) + 8\} - 86m_1^2 m_2^2 m_3^2]$.
..... 47

10607. (W. J. C. Sharp, M.A.)—If Δ_1, Δ_2, Δ_3, Δ_4 be the areas of the faces of the tetrahedron of reference, and (x, y, z, w) the quadruplanar co-ordinates of any point P on $\Delta_1/x + \Delta_2/y + \Delta_3/z + \Delta_4/w = 0$, show that the feet of the perpendiculars from P upon the faces lie in the same plane Find the equation of this plane. 59

10608. (W. J. Greenstreet, M.A.)—If the normal at P on the curve $r^2 = a^2 \cos 2\theta$ meet the axis in H; and if HQ, HR be drawn perpendicular to OH, OP, meeting the circle on the join of the poles of the curves as diameter in Q, R; prove that $HQ : OQ = RP^2 : OQ^2$. 120

10610. (D. Biddle.)—Give a general method of solving Question 9957, already solved in a particular case in Vol. LI., pp. 67, 68. 44

10613. (J. O'Byrne Croke, M.A.)—Prove that the area of an ellipse given by the general equation $(a, b, c, f, g, h)(x, y, 1)^2 = 0$ is $\pi\Delta/C^{\frac{3}{2}}$, where Δ is the discriminant of the equation, and $C = d\Delta/dc$. 84

10616. (S. Tebay, B.A.)—An urn contains florins and half-crowns amounting to £1. 13*s*. If a person draw three coins at random, show that the value of his expectation is $6s. 7\frac{61}{140}d.$ 50

10617. (Rev. H. Sewell, B.A.)—If $a, b, c, \ldots k$ be n positive quantities which are not all equal, prove that, (1)

$$(a^m + b^m + c^m + \ldots + k^m)/n > \{(a + b + c + \ldots + k)/n\}^m;$$

and (2) the general theorem of which the above is a particular case, that $\{a^m + b^m + c^m + \ldots + k^m)/n\}^{1/m}$ increases or decreases with m. 81

10618. (Prof. Brocard.)—Les rayons vecteurs MF, MF′ d'un point d'une ellipse rencontrent la courbe en P, P′; trouver le lieu (1) du pôle de PP′, et du point de rencontre (2) de PP′ avec la tangente menée en M, (3) des perpendiculaires élevées en F sur FM, et en F′ sur F′M.
..... 110

10621. (Professor Sylvester.)—The points of a system containing $2m$ points H and n points O are to be connected together either by single or double bonds, in such a manner that two bonds shall pass through each O, and one through each H. Find the number of distinct species of graphs that can be formed subject to this condition. *Ex.*—Let a, b, c, d and $\alpha, \beta, \gamma, \delta, \epsilon$ be the points H, O; then, denoting any two points x, y with a single bond between them by xy, and with a double bond by $(xy)^2$, we have the following system of conjunction (11 distinct graphs in all):—

$ab, cd, \alpha\beta, \beta\gamma, \gamma\delta, \epsilon\alpha.$	$ab, ca, d\beta, \alpha\gamma, \beta\gamma, (\delta\epsilon)^2$
$ab, cd, \alpha\beta, \beta\gamma, \gamma\delta, (\delta\epsilon)^2$	$a\alpha, ba, c\beta, d\beta, \gamma\delta, \delta\epsilon, \epsilon\gamma$
$ab, ca, da, \beta\gamma, \gamma\delta, \delta\epsilon, \epsilon\beta$	$a\alpha, ba, c\beta, d\gamma, \gamma\delta, \delta\epsilon, \epsilon\beta$
$ab, ca, da, (\beta\gamma)^2, (\delta\epsilon)^2$	$a\alpha, ba, c\beta, d\gamma, \beta\gamma, (\delta\epsilon)^2$
$ab, ca, d\beta, \alpha\beta, \gamma\delta, \delta\epsilon, \epsilon\gamma$	$a\alpha, b\beta, c\gamma, d\delta, \alpha\beta, \gamma\epsilon, \epsilon\delta$
$ab, ca, d\beta, \alpha\gamma, \gamma\delta, \delta\epsilon, \epsilon\beta$	

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10622. (Professor Neuberg.)—Soit K le point de Lemoine du triangle ABC, et soient O, O_a, O_b, O_c les centres des cercles circonscrits aux triangles ABC, KBC, KCA, KAB. Démontrer: (1) que O et K sont les foyers de l'ellipse qui touche les côtés du triangle $O_aO_bO_c$ en leurs milieux; (2) que les quadrilatères OBO_aC, OCO_bA, OAO_cB ont même surface. 96

10623. (Professor Lampe, LL.D.)—Solve, in positive integers,
$$(x-y)^4 = x^3+y^3. \quad \text{..... 94}$$

10637. (Major-General P. O'Connell.)—Solve the equation
$$x^6+378x^5+38189x^4+492368x^3-572554x^2+213720x-26352 = 0.$$
..... 62

10638. (J. Griffiths, M.A.)—If a, b, c, k be all real quantities, and k not greater than unity, prove that $(a+b-2kc)^2 > 4(1-k^2)(ab-c^2)$. 44

10640. (Robert Bryant, D.Sc., B.A.)—The perpendiculars from the foci of an ellipse upon a tangent to the auxiliary circle cut the ellipse in two points on the same side of the axis major. Find geometrically the locus of the intersection of this chord with the aforenamed tangent. 97

10642. (W. J. C. Sharp, M.A.)—If V be the volume of the tetrahedron of reference; Δ_1, Δ_2, Δ_3, Δ_4 the areas of its faces; A′, B′, C′, D′ the projections of a point P, the tetrahedral content coordinates of which are λ', μ', ν', π', upon the faces; show that (1) the content of the tetrahedron PB′C′D′ is $\frac{3^4}{2^2} \cdot \frac{\mu'\nu'\pi'}{\Delta_1^2 \cdot \Delta_2^2 \cdot \Delta_3^2} V^5$; and hence (2) the points A′, B′, C′, D′ all lie in the same plane, if
$$\frac{\Delta_1^2}{\lambda'} + \frac{\Delta_2^2}{\mu'} + \frac{\Delta_3^2}{\nu'} + \frac{\Delta_4^2}{\pi'} = 0. \quad \text{..... 107}$$

10645. (J. O'Byrne Croke, M.A.) — The semi-traces of planes through the axes of x and y of an ellipsoid on the planes of yz, xz, respectively, are r_1 and r_2; if $f_1^2 = a^2-r_1^2$, $f_2^2 = b^2-r_2^2$, $F_1^2 = a^2-c^2$, $F_2^2 = b^2-c^2$; prove that the length of the radius vector, which is half the diameter in which they intersect, is given by
$$\rho^2 = \left\{a^2F_1^2/f_1^2 + b^2F_2^2/f_2^2 - (a^2+b^2-c^2)\right\} / \left\{F_1^2/f_1^2 + F_2^2/f_2^2 - 1\right\}.$$
..... 94

10646. (R. Tucker, M.A.) — If tangents KP, KQ are drawn to a parabola from any point K on the latus rectum, prove geometrically that the orthocentre of the triangle PKQ lies on a straight line parallel to the directrix. 90

10647. (J. J. Barniville, M.A.)—P, Q are the mid-points of the sides BC, CD of a square ABCD; in AP, AM is taken = AB; AQ, MD meet in N: prove that $DN^2 = \frac{5}{9}MN^2$. In AC, AR is taken = AB; DR meets AQ in S: prove that $DS^2 = 2RS^2$. Prove also that, if U, V be the mid-points of AB, AD, then AP and AQ are each mutually trisected by UV and BD; and that consequently (1) UC, BD, PA, and (2) VC, BD, QA, are respectively concurrent. 97

10650. (R. Knowles, B.A.)—If the three diagonals of a hexagon, inscribed in a conic, pass through the some point G; prove that the points of intersection of its opposite sides are on the polar of G. 126

10651. (Prince Camille de Polignac.) — A right line of constant length moves so that three of its points remain on three cylindrical surfaces the generators of which are parallel to a given line. Prove that any other point must remain on a cylindrical surface of the same nature. 87

10653. (Professor Catalan.) — Décomposer, en une somme de deux carrés, le nombre $N = 17^3 . 73^5 = 10\,185\,000\,736\,409$. [See Quest. 9967.] 109

10656. (Professor Emmerich, LL.D.)—Prove that in any triangle, unless it be equilateral, $\cot\omega > \tan\frac{1}{3}A + \tan\frac{1}{3}B + \tan\frac{1}{3}C$. 100

10663. (Professor Villademoros.)—Deux triangles ABC, AB'C' ont le sommet A et la médiane AM communs. Démontrer que, si le côté BC est parallèle à la bissectrice de l'angle B'AC', et tel que $MB^2 = AB' . AC'$, réciproquement le côté B'C' sera parallèle à la bissectrice de l'angle BAC, et tel que $MB'^2 = AB : AC$. 124

10666. (Professor Orchard, M.A.)—A triangle, PQR, right-angled at Q, is inscribed in the conic $4x^2 - 9y^2 - 36 = 0$; show that the envelope of the line PR is the conic $25(4x^2 - 9y^2) = 6084$. 113

10677. (The Rev. T. W. Openshaw.)—Given the vertical angle of a triangle, its altitude, and the sum or difference of the sides; construct the triangle. 107

10678. (J. O'Byrne Croke, M.A.) Show that, if Δ be the discriminant of the conic $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$, and $R^2 = 4h^2 + (a-b)^2$, the semi-axes α, β are given by

$$\alpha^2 = 2\Delta/\{(h^2-ab)(a+b-R)\}, \quad \beta^2 = 2\Delta/\{(h^2-ab)(a+b+R)\}.$$

..... 114

10682. (Professor Cayley, F.R.S.)—If ω be an imaginary cube root of unity, and

$$y = \frac{(\omega-\omega^2)x + \omega^2x^3}{1-\omega^2(\omega-\omega^2)x^2},$$

prove that

$$\frac{dy}{(1-y^2)^{\frac{1}{3}}(1+\omega y^2)^{\frac{1}{3}}} = \frac{(\omega-\omega^2)dx}{(1-x^2)^{\frac{1}{3}}(1+\omega x^2)^{\frac{1}{3}}}.$$ 90

10691. (Professor Hudson, M.A.)—If a face of Cleopatra's needle be a trapezoid having its two parallel sides of lengths 5 ft. 4 in. and 7 ft. 11 in., and their distance apart 61 ft. 6 in., and if it be exposed to a uniform normal pressure of 40 lbs. to the square foot, show that the moment of this pressure about the side 7 ft. 11 in. is $468578\frac{3}{4}$ foot-pound units. 119

10695. (Professor Neuberg.)—Soient P, P' deux points inverses par rapport à un cercle d'Apollonius du triangle ABC. Démontrer que les triangles podaires de ces points par rapport au triangle ABC sont inversement semblables. 105

10697. (Professor de Longchamps.)—Soit BB' le petit axe d'une ellipse Γ; du point B, abstraction faite de BB', on peut mener à Γ deux

normales. Soit C le pied d'une de ces normales. En désignant par F l'un des foyers de Γ, la perpendiculaire élevée en F à BF rencontre la tangente, en B, en un certain point D. Démontrer que BD = BC. 100

10699. (Professor Neuberg.)—Soient A, B, C les centres de trois circonférences qui passent par un même point D et se coupent, en outres deux à deux, aux points A′, B′, C′. Les cordes DA¹, DB¹, DC¹ communes à deux des circonférences rencontrent la troisième circonférence aux points A″, B″, C″. Démontrer que les droites A′A″, B′B″, C′C″ sont doubles des hauteurs du triangle ABC. 102

10709. (H. W. Segar, B.A.)—Solve the equation

$$(x^2-y^2-a^2)\,xdy+(y^2-x^2-a^2)\,ydx=0. \quad \ldots\ldots 99$$

10710. (J. Brill, M.A.)—If A, B, C be the inverses of the points A′, B′, C′ with respect to their centroid O, prove that the product PA . PB . PC is stationary when P coincides with O. 99

10711. (D. Biddle.)—Three weights m_1, m_2, m_3 are situated on three concentric circles, of radius r_1, r_2, r_3 respectively. Find the conditions of equilibrium about the common centre; and particularly give the angular relations when the weights are equal, and the radii as 3, 4, 5. 98

10719. (Professor Hudson, M.A.)—Supposing that the prosperity of a country varies as the excess of the average individual wealth over a given sum, and that by defeat and civil war one-mth of the wealth and one-nth of the population are destroyed, and the nation has to pay a property tax of $100r$ per cent. to the victor: show that the ruling Government must now put to death about 8 per cent. of the original population in order that the prosperity may be the same as before. 106

10724. (Professor Anderson, M.A.)—Perpendiculars are drawn to the symmedians of a triangle, at its angular points, forming another triangle. Show that the circumcentre of the former is the centroid of the latter. 112

10727. (Professor Wolstenholme, M.A., Sc.D.)—If the equation of a conic be $u \equiv (a, b, c, f, g, h \mathbin{\text{XY}} x, y, 1)^2 = 0$,

the equation $4\lambda u = \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2$, where $(\lambda-a)(\lambda-b) = h^2$

is the equation of its directrices..... 122

10732. (J. D. H. Dickson, M.A. Arising out of 10690.)—Find integral values of x, y, z to satisfy the equations

$$x+y+z=0, \quad x^2+y^2+z^2=2\square. \quad \ldots\ldots 116$$

10733. (R. Tucker, M.A.)—A′B′C′, A″B″C″, are circum-triangles of ABC, so drawn that A′B′, A″C″ pass through A and are perpendicular respectively to ΩA Ω′A, &c.; prove that, if Ω_1 (A′B′C′), Ω_1', (A″B″C″) are the other Brocard-points of the circum-triangles, then the four Brocard-points form a rectangle of which the circum-circle is (5) of Quest. 10139 (see *Reprint*, Vol. LI., p. 119) and the Brocard-points (above) are the

Brocard-points of the diametral triangle of ABC. [For general properties of the above triangles, see SIMMONS's *Recent Geometry*, p. 178.] 104

10737. (R. F. Davis, M.A.)—Give a *geometrical* proof of the formula $\cot\omega = \cot A + \cot B + \cot C$. 108

10747. (Rev. T. C. Simmons, M.A.)—A triangle is formed by three straight lines chosen at random. Show that the chance of its being acute-angled is $\frac{1}{4}(4-\pi) = \cdot 4292$ nearly. 108

10762. (Professor Blater, LL.D.)—Prove that, if the first five decimal places of the reciprocal of any number be given, the next five places may readily be found by the use of a Table of Quarter Squares 122

10776. (R. Tucker, M.A.)—P, Q, R are three co-normal points on a parabola. The tangents at these points cut any double ordinate in L, M, N; the other tangents through L, M, N intersect in p, q, r. Prove that the circle pqr passes through the vertex. 121

10786. (J. Brill, M.A.)—The moments of n complanar forces about a point O in their plane are $m_1, m_2, \ldots, m_n$. Through O a straight line is drawn meeting the lines of action of the forces in the points $r_1, r_2, \ldots, r_n$, and a point R is taken on this line so that

$$\frac{m_1+m_2+\ldots+m_n}{OR} = \frac{m_1}{Or_1}+\frac{m_2}{Or_2}+\ldots+\frac{m_n}{Or_n};$$

prove that the locus of R is the line of action of the resultant of the given forces. 125

10791. (Rev. T. C. Simmons, M.A.)—Prove that the mean value of the nth power of the distance between two points taken at random within a given circle, n being not less than -1, is

$$\frac{2^{n+4}}{(n+2)(n+3)\pi^{\frac{1}{2}}}\cdot\frac{\Gamma[\frac{1}{2}(n+5)]}{\Gamma[\frac{1}{2}(n+6)]}r^n$$ 119

10806. (Professor Wolstenholme, M.A., Sc.D.)—If x, y, z be three finite quantities, such that $x+y+z=0$.....(1),

$(ax+by+cz)^2 = 4(bc+ca+ab)(yz+zx+xy)$, $ax^2+by^2+cz^2=0$...(2, 3),

prove that $(a+b+c)(bc+ca+ab) = 9abc$. 126

APPENDIX I.

Elementary Geometrical Note on the "Frégier-Point" of an Ellipse. By R. F. DAVIS, M.A. 129

APPENDIX II.

Unsolved Questions 132

CORRIGENDA.

Page 108, line 6 from bottom, *for* 10717 *read* 10747;

Page 111, line 1, *for* 10194 *read* 10494;

Page 117, line 15, *for* 10156 *read* 10456;

Page 117, last line, *for* pp. 31, 2 *read* pp. 81–2.

MATHEMATICS

FROM

THE EDUCATIONAL TIMES.

WITH ADDITIONAL PAPERS AND SOLUTIONS.

10447. (Professor DE LONGCHAMPS.)—Soient une circonférence Δ et deux diamètres rectangulaires Δ', Δ''. D'un point M, de Δ, on abaisse des perpendiculaires MP', MP'' sur ces diamètres : soient P le pôle de la droite P'P'' par rapport à Δ; Q, R, ses projections sur Δ', Δ''. Démontrer que QR est tangente à Δ.

Solution by D. BIDDLE; G. G. MORRICE, M.B.; *and others.*

Let $\angle MOQ \equiv \alpha$; then $OP' = \cos\alpha$,

$\angle OP'P'' = \alpha$, $OI = \sin\alpha\cos\alpha$;

Again, $\angle ROP = \alpha$,

and $OP = 1/(\sin\alpha \,.\, \cos\alpha)$,

the reciprocal of OI. Also,

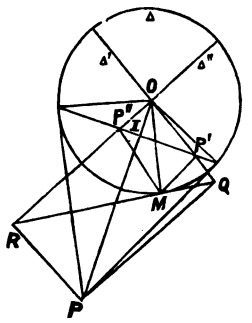
$OR = OP \,.\, \cos\alpha = \operatorname{cosec}\alpha$.

Similarly, $OQ = \sec\alpha$;

and $RQ = OP = 1/(\sin\alpha \,.\, \cos\alpha)$

$= \tan\alpha + \cot\alpha$;

therefore QR is tangential to Δ in M.

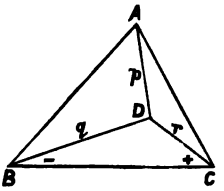


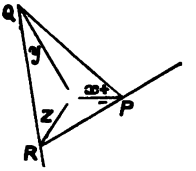
[PQ is evidently the polar of P'', and PR of P'; hence Q, R are, respectively, the poles of P''M, P'M; consequently QR is the polar of M, or, in other words, the tangent at M.]

10556. (Professor NEUBERG.)—Quatre forces parallèles, dont les points d'application sont situés dans un même plan, se font équilibre. On applique chacune d'elles au centre du cercle passant par les points d'application des trois autres. La direction des quatre forces restant la même, démontrer que l'équilibre existe également dans la nouvelle position.

Solution by Professors GENESE, ANDERSON, *and others.*

Let p, q, r be the distances of the point D from A, B, C. Let PQR be a triangle with its sides parallel to p, q, r; and from PQR let x, y, z be drawn parallel to BC, CA, AB. Then the six angles at P, Q, R are seen to be equal to those at A, B, C in the same cyclical order, but not





correspondingly situated. Thus x, y, z cointersect, at O say.

Moreover, the triangles about O are proportional to those about D. For

$$\frac{\text{ORP}}{\text{OPQ}} = \frac{zx\sin\widehat{zx}}{xy\sin\widehat{xy}} = \frac{\sin\text{OQR}\sin\widehat{zx}}{\sin\widehat{\text{ORQ}}\sin\widehat{xy}} = \frac{\sin\text{DAC}\sin\text{ABC}}{\sin\text{DAB}\sin\text{BCA}}$$

$$= \frac{\sin\text{DAC}\,.\,b}{\sin\text{DAB}\,.\,c} = \frac{\text{DCA}}{\text{DAB}}, \text{ \&c.}$$

Thus, if D be the centre of gravity of m_1, m_2, m_3 at A, B, C, O will be that of m_1, m_2, m_3 at P, Q, R. It will be seen immediately that the centres in the theorem have the same *relative* disposition as O, P, Q, R.

7887. (Professor MUKHOPÂDHYÂY.)—Express, in finite form,

$$\text{S} \equiv \frac{x}{1^2} + \frac{x^3}{3^2} + \frac{x^5}{5^2} + \ldots \frac{x^{2n-1}}{(2n-1)^2} + \ldots \textit{ ad inf.}$$

Solution by D. EDWARDES; R. KNOWLES, B.A.; *and others.*

The series in question is convergent if x is less than unity, and then $\tan^{-1}x^{\frac{1}{2}} = x^{\frac{1}{2}} - \frac{1}{3}x^{\frac{3}{2}} + \frac{1}{5}x^{\frac{5}{2}} - \text{\&c.}$; thus let $u = x^{\frac{1}{2}}e^{i\phi}$, $v = x^{\frac{1}{2}}e^{-i\phi}$; then

$$\tan^{-1}u + \tan^{-1}v = \tan^{-1}\frac{2x^{\frac{1}{2}}\cos\phi}{1-x}$$

$$= 2\left\{x^{\frac{1}{2}}\cos\phi - \tfrac{1}{3}x^{\frac{3}{2}}\cos 3\phi + \tfrac{1}{5}x^{\frac{5}{2}}\cos 5\phi - \ldots\right\}.$$

Therefore $$\int_0^\pi \left(\tan^{-1}\frac{2x^{\frac{1}{2}}\cos\phi}{1-x}\right)^2 d\phi = 2\int_0^{\frac{1}{2}\pi}\left(\tan^{-1}\frac{2x^{\frac{1}{2}}\cos\phi}{1-x}\right)^2 d\phi$$

$$= 4\int_0^\pi \left\{x\cos^2\phi + \frac{1}{3^2}x^3\cos^2 3\phi + \frac{1}{5^2}x^5\cos^2 5\phi + \ldots\right\} d\phi,$$

the terms involving products of cosines vanishing, when integrated; or, since $\int_0^{\pi} \cos^2 r\phi \, d\phi = \frac{1}{2}\pi$,

$$\int_0^{\frac{1}{2}\pi} \left(\tan^{-1} \frac{2x^{\frac{1}{2}} \cos\phi}{1-x} \right)^2 d\phi = \pi \left(x + \frac{x^3}{3^2} + \frac{x^5}{5^2} + \dots \text{ ad inf.} \right).$$

Otherwise $$S = \int_0^x \log \left(\frac{1+x}{1-x} \right)^{-2x} dx.$$

10124. (D. Biddle.)—A cubical die, of side a, lies at the bottom of a cylindrical dice-box, of radius r. The dots or pips being symmetrically arranged on their several sides and each of radius ρ, find the probability that one of those on the uppermost side of the cube is placed vertically over the centre of the box.

Solution by John J. Barniville.

When the ace is uppermost, the radius of the circle within which it must lie is $\{r^2 + \rho^2 - a\rho + (2\rho - a)(r^2 - \frac{1}{4}a^2)^{\frac{1}{2}}\}^{\frac{1}{2}}$.

Hence $$P_1 = \rho^2 \div \{r^2 + \rho^2 - a\rho + (2\rho - a)(r^2 - \tfrac{1}{4}a^2)^{\frac{1}{2}}\}.$$

In like manner $$P_2 = \frac{2\rho^2}{r^2}, \quad P_3 = \frac{2\rho^2}{r^2} + \frac{\rho^2}{r^2 - \frac{1}{4}a^2},$$

$$P_4 = \frac{4\rho^2}{r^2}, \quad P_5 = \frac{4\rho^2}{r^2} + P_1, \quad P_6 = \frac{4\rho^2}{r^2} + \frac{2\rho^2}{r^2 - \frac{1}{4}a^2}.$$

Hence the probability is $\frac{1}{3}P_1 + \frac{8}{3} \cdot \frac{\rho^2}{r^2} + \frac{2\rho^2}{4r^2 - a^2}$.

9031. (Professor Pramathanath Dather, M.A.)—Trace the curve $xy^2 + x^3 + ay^2 = 0$, and find its first negative pedal with respect to the origin.

Solution by Professor Schoute.

When the x is taken with opposite sign, the equation is $y^2(a-x) - x^3 = 0$, and the curve the Cissoid in the ordinary position (*Leçons de géométrie analytique*, Briot et Bouquet, 13me édition, page 15).

We put $y = \mu x$, and find x and y expressed in μ by the formulas

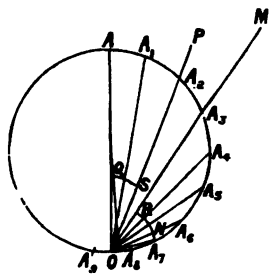
$$x = \frac{a\mu^2}{1+\mu^2}, \quad y = \frac{a\mu^3}{1+\mu^2}.$$

Now the first negative pedal for O is the envelope of one side of a right angle, whose other side passes through O, and whose vertex describes the curve. But the locus of the mid-points of its radii vectores from O is also the locus of the centres of the circles through O that touch the curve. As the circle $x^2+y^2-\alpha x-\beta y=0$ touches the cissoid, when the equation $a\mu^2-\beta\mu-\alpha=0$ obtained by the substitution of the values of x and y has two equal roots, the first negative pedal is the parabola $\beta^2-4a\alpha=0$. This result is not new. For it is known that the first positive pedal of the parabola $y^2-2ax=0$ is the cissoid $y^2(a-2x)-2x^3=0$ (*Traité de géométrie analytique*, Picquet).

10176. (B. F. Finkel, B.Sc.)—Inscribe a regular 17-gon in a circle.

Solution by J. J. Barniville; Professor Ignacio Beyens; *and others.*

Analysis.—Let A, A_1, A_2, &c. be the angular points, AO the diameter. Produce OA_2 to M and OA_2 to P, making $A_2M=OA_5$, and $A_2P=OA_8$. Again, cut off $A_6N=OA_7$, and $A_1Q=OA_4$. Lastly, cut off OR = ON, and OS = OQ.



Then $\rho_1\rho_4=R(\rho_3+\rho_5)=R.OM$,

$\rho_2\rho_6=R(\rho_6-\rho_7)=R.ON$.

Denoting the supplementary chords of ρ_1, &c. by r_1, &c., we have

$r_1\rho_1=Rr_2$, $r_2\rho_2=Rr_4$, &c.;

hence $\rho_1\rho_2\rho_4\rho_8=R^4$, and $OM.ON=R^2$.

In like manner, $OP.OQ=R^2$. Again,

$$\begin{aligned}OM.ON&=(\rho_3+\rho_5)(\rho_6-\rho_7)=\rho_3\rho_6+\rho_5\rho_6-\rho_3\rho_7-\rho_5\rho_7\\&=R(\rho_3-\rho_8)+R(\rho_1-\rho_6)-R(\rho_2-\rho_7)-R(\rho_2-\rho_5)\\&=R(OM-ON-OP+OQ)=R(MR-PS);\end{aligned}$$

therefore $MR-PS=R$.

Again, $MR.PS=(OM-ON)(OP-OQ)$

$=(\rho_3+\rho_5-\rho_6+\rho_7)(\rho_2+\rho_8-\rho_1+\rho_4)$;

therefore $MR.PS=4R(OM-ON-OP+OQ)=4R^2$.

Hence MR (and PS) are given, and OM − ON is given; but $OM.ON=R^2$, therefore OM and ON are given. Similarly, OP and OQ are given; but $\rho_6\rho_7=R(\rho_1-\rho_4)=R.OQ$, and $\rho_6-\rho_7=ON$; hence ρ_6 and ρ_7 are given, hence the arc A_6A_7 is given.

Summary. — $OM.ON=OP.OQ=R^2$, $OM.ON=R(MR-PS)$; hence $MR-PS=R$. Hence we have given MR, PS, OM, ON, OP, OQ, ρ_6, ρ_7, A_6A_7.

4160. (C. H. Hinton, M.A.)—If there be two circles cut by a straight line, and the points where each circle is cut be joined to its centre and the

joining line produced,—(1) show that the four triangles thus formed on portions of the straight line as bases have their sides all in the same ratio; (2) how must the straight line move that the triangles thus formed have their sides in the same ratio for every position.

Solution by D. Biddle *and* Morgan Brierley.

There are two cases under this question, viz., where the given line intersects the junction of the centres at a point P between them (Fig. 1),

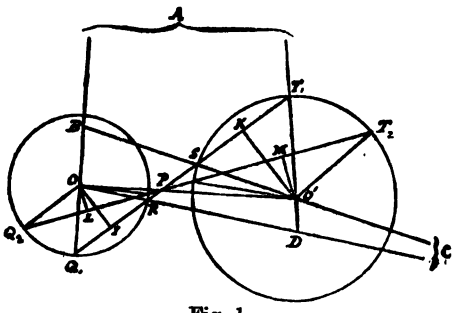


Fig. 1.

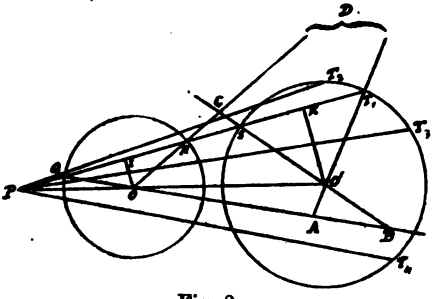


Fig. 2.

and where it intersects it at a point P beyond (Fig. 2). In either case the four triangles are QTA, QSB, RSC, RTD.

(1) The angles at the base of the several triangles are the same or supplementary, and it follows that the sides, which are as the sines of the opposite angles, are in the same ratio.

(2) This ratio will be preserved for any new series of triangles, provided the base line pass through the same point P, because the ratio of the radii (OQ : O'T) is constant, and OP : O'P is also constant, and it

follows that the sines of the angles formed with the moving line by the radii of the two circles, remain in the same ratio. Thus (Fig. 1)

$$OL : O'M = OI : O'K,$$

whence $\sin OQ_2L : \sin O'T_2M = \sin OQ_1I : \sin O'T_1K$;

and the same holds in Fig. 2, if perpendiculars be drawn from O, O′ to the moving line.

10492. (Professor NEUBERG.) — Cinq forces parallèles se faisant équilibre, on transporte chacune d'elles au centre de la sphère passant par les quatre autres points d'application. Démontrer que l'équilibre existe encore, pourvu que les forces restent parallèles entre elles.

Solution by Professors ANDERSON, MUKHOPÂDHYÂY, *and others.*

Let A, B, C, D, E be the points of application of the five forces, and *a*, *b*, *c*, *d*, *e* the centres of spheres circumscribing the tetrahedra BCDE, CDEA, DEAB, EABC, ABCD. Let EB meet the plane ADC in the point N, and join AN. Also let *eb* meet the plane *adc* in *n*, and through the point *n* draw a plane perpendicular to *cd*, meeting this line in *k*, and intersecting the planes *edc*, *bdc* in the lines *lk*, *mk*. Lastly, through *be* draw a plane parallel to the line *cd*, meeting the lines *lk*, *mk* in the points *l* and *m*.

The $\angle lkn$ is the angle between the planes *adc* and *edc*, and is therefore equal or supplemental to $\angle ABE$. Similarly, $\angle mkn$ is equal or supplemental to $\angle AEB$. Again, AN is the line of intersection of the planes ADC, ABE, and is therefore perpendicular to the plane through *be* parallel to *cd*. Also EA is perpendicular to the plane *bcd*. Hence $\angle EAN$ is equal or supplemental to $\angle lmk$; and, in like manner, $\angle BAN$ is equal or supplemental to $\angle mlk$.

Hence (as in solution of 10556) $\dfrac{ln}{nm} = \dfrac{EN}{NB}$.

But plainly $\dfrac{ln}{nm} = \dfrac{en}{nb}$, and therefore $\dfrac{EN}{NB} = \dfrac{en}{nb}$.

If now P, Q, R, S, T denote the parallel forces whose points of application are A, B, C, D, E, the necessary and sufficient conditions of equilibrium are $P+Q+R+S+T = 0$, $Q.BN - T.EN = 0$,

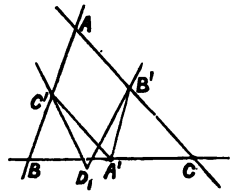
with two other equations similar to the latter. When the forces are transferred to the points *a*, *b*, *c*, *d*, *e*, we have

$$P+Q+R+S+T = 0, \quad Q.bn - T.en = 0,$$

with two others, and therefore the conditions for equilibrium are satisfied.

10211. (E. LEMOINE.)—Par les milieux A′, B′, C′ des côtés d'un triangle quelconque ABC, on mène les symétriques de BC, CA, AB par rapport à une direction donnée. Ces trois droites concourent en un point du cercle des neuf points du triangle ABC.

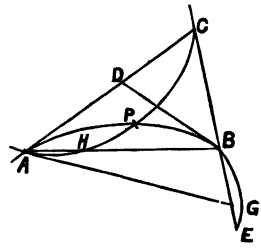
Solution by J. J. BARNIVILLE; Professor ARMENANTE; *and others.*



Let D_1, D_2, D_3 be the intersections; then angle $C'D_1B' = C'AB' = C'A'B'$, therefore D_1 lies on the circle $A'B'C'$. And, since D_2 lies on the line $C'D_1$ and on the circle $A'B'C'$, it evidently coincides with D_1; so also for D_3.

10436. (Professor DELAHAYE.)—Mener par les sommets d'un triangle trois droites concourantes déterminant sur les côtés trois segments non consécutifs égaux entre aux.

Solution by Prof. ANDERSON, M.A.; H. J. WOODALL; *and others.*



The length of the segment can, of course, be found by the solution of the equation

$$(a-x)(b-x)(c-x)-x^3 = 0,$$

a, b, c being the sides of the triangle.

If $a > b > c$, there is only one real root which is positive and $< c$. Hence there is always one, and only one, point satisfying the required condition, and it lies inside the triangle.

To obtain a geometrical solution, we shall first find the locus of the point of intersection of the two straight lines BM and AN drawn from the vertices B and A, and meeting the opposite sides in two points M and N such that AM = CN. Taking the point A for origin, and the sides AB, AC for axes of x and y, the equation of this locus is

$$y^2 + \frac{ab}{c}x\left(\frac{x}{c}+\frac{y}{a}-1\right) = 0.$$

If AD be taken equal in length to a, and CE equal to b, the above conic passes through the point E and touches AD and BD at the points A and B. This conic can easily be constructed. Now, take BH = a, and CG = c, and construct another conic passing through H and touching the lines AG and CG at the points A and C respectively; the point of intersection P of these two conics is the required point.

10543. (R. TUCKER, M.A.)—Eliminate t from the equations

$$2t^4-t^2+1)x+(2t^5-2t^3+t)y = 4at^4, \quad (t^4-t^2+2)tx-(t^4-2t^2+2)y = 4at.$$

Solution by the PROPOSER.

(i.) + t (ii.) gives $ty(t^2-1)+x(t^4+1)=4at^2$.....(iii.),

t (i.) − (ii.) gives $y(2t^4-3t^2+2)+tx(t^2-1)=4at(t^2-1)$.....(iv.).

Eliminating x and y from (iii.), (iv.), we have

$$y(t^4-t^2+1)=2at(t^2-1),\quad x(t^4-t^2+1)=2at^2 \text{ ..... (v., vi.),}$$

whence $$y/x=(t^2-1)/t=t-1/t=z.$$

Now (v.) and (vi.) may be written $y(z^2+1)=2az$, $x(z^2+1)=2a$, whence (vii.) $$x^2+y^2=2ax.$$

If AP, AQ are orthogonal vertical vectors of a parabola, then, if t is the tangent of PAX, (i.), (ii.) are the equations to the "S" lines QL, PL′ of the triangle APQ; hence the locus of the "S" point of APQ is the circle in (vii.), which is immediately seen (geometrically) from the fact that PQ cuts the axis in a point K, such that AK = 4AS.

10466. (R. H. W. WHAPHAM, B.A.)—The two tangents from $x'y'$ to $x^2/a^2+y^2/b^2=1$ form with $lx+my+n=0$ a triangle circumscribing an ellipse; show that, if ξ, η is the middle point of $lx+my+n=0$, we have

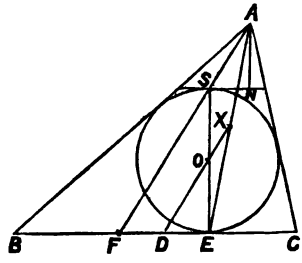
$$\xi : nx'-la^2 = \eta : ny'-mb^2 = -1 : lx'+my'-n.$$

Solution by Professors ANDERSON, AIYAR, *and others.*

Draw the inscribed circle of the triangle ABC, and join the middle point of the base D to X the middle point of AE. Then it is easily proved that DX passes through O the centre of the circle. Also, if EO meet the circle again on S, AS is parallel to DX and equal to 2 . OX.

Hence, if AN be drawn perpendicular to the tangent at S,

$$AN : OE :: 2 . OX : OD.$$



The result in the question may be obtained by projection from these. Let α, β be the coordinates of the point of contact of $lx+my+n=0$, which is the projection of BC; then

$$\alpha=-\frac{a^2l}{n} \text{ and } \beta=-\frac{b^2m}{n}.$$

Again, the equation of the line into which SN is projected is

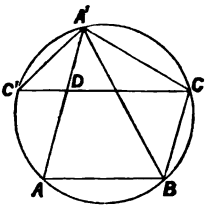
$$lx+my-n=0 \text{ and } AN/OS=(lx'+my'-n)/n$$

also, $$\frac{2 . OX}{OD}=-\frac{x'+\alpha}{\xi}=-\frac{y'+\beta}{\eta}.$$

Hence $$\frac{x'+\alpha}{\xi} = \frac{y'+\beta}{\eta} = \frac{lx'+my'-n}{-n},$$

and, substituting for α and β, we have the result.

10505. (R. Knowles, B.A. Suggested by Quest. 10385.)—A, B, C, D, taken in order, are the angular points of a parallelogram; the circle ABC cuts AD in A′, and DC in C′; prove that A′D : A′C′ = A′C : A′B.



Solution by Professor Zerr; Rev. J. L. Kitchin, M.A.; *and others.*

$\angle$ A′C′C = A′BC, $\angle$ C′A′A = BA′C;

hence the triangles A′C′D, A′BC are similar;

therefore A′D : A′C′ = A′C : A′B.

10095. (Rev. J. J. Milne, M.A.)—If G is the centroid of any triangle, prove that (1)

$$\cot GAB + \cot GBC + \cot GCA = 3\cot\omega = \cot ABG + \cot BCG + \cot CAG,$$

$$\cot AGB + \cot BGC + \cot GCA + \cot\omega = 0, \quad \cot\omega = \cot A + \cot B + \cot C;$$

(2) if K be the symmedian point, that is, a point within the triangle such that $\angle$ KAC = $\angle$ GAB, and two similar relations, then

$$\cot AKB + \cot BKC + \cot CKA + \tfrac{1}{2}\cot\omega + \tfrac{3}{2}\tan\omega = 0.$$

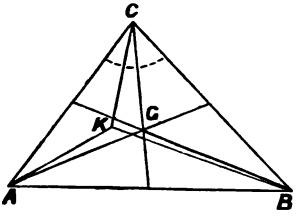
Solution by R. Knowles, B.A.; Rev. J. L. Kitchin; *and others.*

The coordinates of G are proportional to $\frac{1}{3}a$, &c., and

$$GA = \tfrac{1}{3}(2b^2+2c^2-a^2)^{\frac{1}{2}}, \text{ \&c.};$$

$$\cot GAB = (3c^2+b^2-a^2)/4\Delta, \text{ \&c.};$$

$$\therefore \cot GAB + \cot GBC + \cot GCA = 3(a^2+b^2+c^2)/4\Delta = 3\cot\omega = \cot ABG + \cot BCG + \cot CAG.$$



From the triangles AGB, &c., we find

$$\cot AGB = (a^2+b^2-5c^2)/12\Delta, \text{ \&c.};$$

and $$\cot AGB + \cot BGC + \cot AGC + \cot\omega = -(a^2+b^2+c^2)/4\Delta + \cot\omega = 0;$$

$$\cot A + \cot B + \cot C = \cot B + b^2/2\Delta = (b^2+ac\cos B)/2\Delta = (a^2+b^2+c^2)/4\Delta = \cot\omega.$$

2. $$AK = bc(2b^2+2c^2-a^2)^{\frac{1}{2}}/(a^2+b^2+c^2), \text{ \&c.};$$
$$\cot AKB = (b^4-4a^2b^2+a^4-c^4)/4\Delta\,(a^2+b^2+c^2), \text{ \&c.};$$
$$\therefore \quad \cot AKB + \cot BKC + \cot CKA + \tfrac{1}{4}\cot\omega + \tfrac{3}{4}\tan\omega$$
$$= (3a^4+3b^4+3c^4-6a^2b^2-6b^2c^2\cos^2 A - 6a^2c^2\cos^2 B)/8\Delta\,(a^2+b^2+c^2) = 0.$$

10276. (Rev. T. R. Terry, M.A., F.R.A.S.)—If $gh = V^2\cot^2\alpha\log\sec\alpha$, where V, g, and α are constants, and $p = 2gxV^{-2}\tan^2\alpha$, find the value of the integrals

$$I = \frac{\sin\alpha}{V}\int_0^h \frac{dx}{\sqrt{e^{-p}-\cos^2\alpha}} \text{ and } I' = \frac{\sin\alpha}{V\cos^2\alpha}\int_0^h \frac{dx}{\sqrt{\sec^2\alpha - e^p}}.$$

Solution by the Proposer.

Changing the variable from x to p, we have

$$I = \frac{V\cos^2\alpha}{2g\sin\alpha}\int \frac{dp}{\sqrt{e^{-p}-\cos^2\alpha}} \text{ and } I' = \frac{V}{2g\sin\alpha}\int \frac{dp}{\sqrt{\sec^2\alpha - e^p}},$$

the limits being 0 to $\log\sec^2\alpha$.

Putting $\cos^2\alpha\, e^p = \cos^2\theta$, we get

$$I = Vg^{-1}\cot\alpha\int_0^\alpha d\theta = Vg^{-1}\alpha\cot\alpha.$$

$$I' = Vg^{-1}\cot\alpha\int_0^\alpha \sec\theta\, d\theta = \tfrac{1}{2}Vg^{-1}\cot\alpha\log\frac{1+\sin\alpha}{1-\sin\alpha}.$$

Note on Questions **9812, 9813.** *By* Professor Lampe, LL.D.

The equivalent developments may be found in Schellbach's *Die Lehre von den Elliptischen Integralen und den Thetafunctionen* (Berlin, Georg Reimer, 1864, § 36, &c., p. 53, &c.). Compare also Faà de Bruno, in *American Journal, V.*, pp. 1–25. I hinted at these expressions in a note, inserted in the *Verhandlungen* of the Physical Society of Berlin, Vol. VII., p. 47–52 (1888), where I added several problems of the Analytical Mechanics to be solved by these formulæ. [For these Questions, see Appendix II. to Vol. XLIX., or the Contents to this Volume.]

10002. (Artemas Martin, LL.D.)—A speaks the truth x times out of y. We do not know the values of either x or y, but x may be 0, 1, 2, 3, 4, 5, ... y; and y may be 1, 2, 3, 4, ... n. A asserts the occurrence of an event about which we know nothing; find the probability of the truth of A's statement.

Solution by D. BIDDLE.

Under the conditions, we have no more ground for believing, than for disbelieving, A's statement. Thus, let X = the mean value of x, in terms of y; then, since 0 lies at the beginning of every series of values, we have

$$X = \frac{1}{y+1} \cdot \frac{y(y+1)}{2} = \frac{y}{2},$$

and (whatever n and y may be) $P = X/y = \frac{1}{2}$.

10473. (Colonel SERVAIS.)—Si m est un nombre entier, on a

$$\frac{1 \cdot 3 \cdot 5 \ldots (2m+1)}{2 \cdot 4 \cdot 6 \ldots (2m+2)} = \frac{1}{2} - m\frac{1}{2 \cdot 4} + \frac{m(m-1)}{1 \cdot 2} \cdot \frac{1 \cdot 3}{2 \cdot 4 \cdot 6} - \ldots \quad \ldots(1),$$

$$\frac{2 \cdot 4 \cdot 6 \ldots 2m}{3 \cdot 5 \cdot 7 \ldots (2m+1)} = 1 - m\frac{1}{3} + \frac{m(m-1)}{1 \cdot 2} \cdot \frac{1}{5} - \ldots \quad \ldots\ldots\ldots(2).$$

Solution by Rev. T. R. TERRY, M.A.; M. F. EVANS, M.A.; *and others.*

By the formula $F\{\alpha, \beta, \gamma, 1\} = \dfrac{\Gamma(\gamma) \cdot \Gamma(\gamma-\alpha-\beta)}{\Gamma(\gamma-\alpha) \cdot \Gamma(\gamma-\beta)}$,

we have (1) $= \frac{1}{2}F\{-m, \frac{1}{2}, 2, 1\} = \frac{1}{2}\dfrac{\Gamma(2) \cdot \Gamma(m+\frac{3}{2})}{\Gamma(\frac{3}{2}) \cdot \Gamma(m+2)}$ = required result.

$$(2) = F\{-m, \tfrac{1}{2}, \tfrac{3}{2}, 1\} = \frac{\Gamma(\frac{3}{2}) \cdot \Gamma(m+1)}{\Gamma(1) \cdot \Gamma(m+\frac{3}{2})} = \text{required result.}$$

[(1) is proved by equating the coefficients of x^{m+1} in the identity

$$(1+x)^{m+\frac{1}{2}} = (1+x)^m \cdot (1+x)^{\frac{1}{2}}.$$

The right-hand member of (2)

$$= \int_0^1 (1-x^2)^m \, dx = \int_0^{\frac{1}{2}\pi} \sin^{2m+1}\theta \, d\theta, \quad \text{if } x = \cos\theta;$$

$$= \ldots\ldots \text{ (WALLIS' Formula).]}$$

3973. (Professor HUDSON, M.A.)—A flat circular disc of radius a is projected upon a rough horizontal table, which is such that the friction upon an element a is cV^2ma, where V is the velocity of the element, m the mass of a unit of area: find the path of the centre of the disc.

If the initial velocity of the centre of gravity and the angular velocity of the disc be u_0, ω_0, prove that the velocity u and angular velocity ω, at any subsequent time, satisfy the equation $\left(\dfrac{3u^2-a^2\omega^2}{3u_0^2-a^2\omega_0^2}\right)^2 = \dfrac{u^2\omega}{u_0^2\omega_0}$.

Solution by Professor ANDERSON, M.A.

Let u and v be the velocities of the centre of the disc resolved in two fixed directions at right angles, and ω the angular velocity. Then, referring any point P of the disc to axes through its centre parallel to the fixed directions, the resolved velocities of P are

$$u-\omega y \quad \text{and} \quad v+\omega x, \quad \text{and} \quad V^2 = u^2+v^2-2u\omega y+2v\omega x+\omega^2 r^2.$$

The friction on the element at P acts in the direction opposite to that of its motion, and its components are

$$mX = -cm\alpha V^2(u-\omega y), \qquad mY = -cm\alpha V^2(v+\omega x);$$

therefore $\Sigma mX = -c\Sigma m\alpha\,(u-\omega y)(u^2+v^2-2u\omega y+2v\omega x+\omega^2 r^2)$,

which, neglecting terms which vanish, becomes

$$-cMu\,(u^2+v^2+a^2\omega^2), \quad \text{M being the mass of the disc.}$$

In like manner, $\Sigma mY = -cMv\,(u^2+v^2+a^2\omega^2)$.

Hence $\dfrac{\dot{u}}{\dot{v}} = \dfrac{u}{v}$, and $\dfrac{u}{v} = \text{const.}$ The path is therefore a straight line, and we can take u as the velocity in direction and magnitude, and make $v = 0$; therefore $\dot{u} = -cu(u^2+a^2\omega^2)$ .....(1).

Taking, now, moments for the frictional forces about the centre of gravity, we have $M\tfrac{1}{4}a^2\dot{\omega} = c\Sigma m\alpha\,[V^2(u-\omega y)\,y - V^2\omega x^2]$,

which reduces to $-cMa^2u^2\omega - cM\tfrac{1}{4}\omega^3 a^4$;

therefore $\dot{\omega} = -2cu^2\omega - \tfrac{1}{4}ca^2\omega^3$.....(2).

From equations (1) and (2), we get

$$\frac{d\omega}{du} = \frac{2\,(3u^2\omega+a^2\omega^3)}{3\,(u^3+a^2u\omega^2)},$$

a homogeneous equation very easily solved, and giving at once the result in the question.

10557. (Professor NILKANTA SARKAR.)—An electric cable is laid between two ports, and it is found that the gross annual receipts amount to 20 per cent. on the original outlay : the annual cost of repairs is 2 per cent. on the original cost, and the expense of working is found to be £600 a year in addition to 20 per cent. on the net receipts. After a year, a second cable is laid down, and it is found that the gross receipts and the expense of working are doubled, while the cost of repairs for the new cable is only 1 per cent., and that the *net* receipts now amount to £14,500. Find the cost of a cable and the net receipts for the first year.

Solution by W. J. GREENSTREET, M.A. ; Prof. ZERR, M.A. ; *and others.*

Let G = gross receipts ; £x = cost of cable ; R = cost of repairs ; W = working expenses,

$$G = \frac{x}{5}, \quad R = \frac{x}{50}, \quad W = \frac{5}{6}\left(600+\frac{9x}{250}\right);$$

and, denoting the next year's expenditure, &c. by dashes,

$$G' = \frac{2x}{5}, \quad R' = \frac{3x}{100}, \quad W' = \frac{10}{6}\left(600 + \frac{9x}{250}\right),$$

and

$$14500 = \frac{40x - 3x - 6x}{100} - 1000;$$

whence $x = £50{,}000$, and net receipts = £7000.

10088 & 10122. (F. R. J. Hervey.)—(10088.) Prove that, if, for each of the four triangles formed by four lines, a line be drawn bisecting perpendicularly the distance from circumcentre to orthocentre, the four bisecting lines are concurrent.

(10122.) Supposing the nine-point circle of a triangle to vary so that its radius retains a constant ratio to the distances of its centre from circumcentre and orthocentre, show that it will always meet the lines of the triangle at the vertices of two triangles similar to the given and pedal triangles respectively.

Solution by the Proposer.

Suppose three tangents to a three-cusped hypocycloid to revolve together, forming a variable triangle ABC of constant shape, having circumcentre K and orthocentre O. A position will be arrived at in which the vertex circle S of the cycloid becomes the nine-point circle of the triangle. [See Solution of 9525, Vol. lii.] Let the intersections on S which are then mid-points of sides be a, b, c, and the others p, q, r. The triangles abc and pqr simply rotate. For convenience, let the cycloid rotate so that abc is fixed. Then K is fixed, and A, B, C describe circles having KA, KB, KC (of above position) for diameters; hence, O describes a circle having KO for diameter. Thus, the centre of the hypocycloid is always equally distant from the circumcentre and orthocentre of a tangent triangle, and this distance is constant for a triangle of given species. Questions 10088 and 10122 follow; 10122 by introducing rotation and expansion so as to make ABC the fixed triangle, in place of abc; and 10088, since (by 10015, Vol. lii.) a three-cusped hypocycloid can generally be determined so as to touch four given lines.

3557 & 3615. (J. B. Sanders.)—Find the velocity and angle of elevation of a ball in order that it may be 100 feet above the ground at the distance of one mile. If the ball is projected from the top of a tower 200 feet high, at an angle of elevation of 60°, with a velocity of 50 feet, find the range on the horizontal plane passing through the foot of the tower, and the time of flight.

Solution by Professor ZERR, M.A., *and* G. G. STORR, M.A.

(1) Let α, v be the angle and velocity of projection; then the horizontal range $= v^2/g \sin 2\alpha = 5280$ feet, and the equation to the path of the ball, the point of projection being origin, is $y = x \tan \alpha - gx^2/(2v^2 \cos^2 \alpha)$. Putting $x = 1320$ ($\frac{1}{4}$ mile), and $y = 100$, $g = 32$, we have $\tan \alpha = \frac{19}{33}$ or $\alpha = 5° 46' 4''$, $v = 919{\cdot}176$ feet per second.

(2) Putting $y = -200$, $\alpha = 60$, $v = 50$, we have $x = 127{\cdot}9$ feet, and $t = \dfrac{x}{v \cos \alpha} = 5{\cdot}1$ seconds.

10482. (Prof. ZERR.)—Give a formula for differentiating $y = x^{x^{x^{\&c.}}}$ and show that the same formula will answer for $y = a^{x^{x^{\&c.}}}$ and $y = e^{x^{x^{\&c.}}}$

Solution by H. J. WOODALL; Rev. J. L. KITCHIN; *and others.*

1. $y = x^{x^{x^{\&c.}}} = x^y$, therefore $\log_e y = y \log_e x$,

$$\frac{1}{y}\frac{dy}{dx} = \frac{dy}{dx}\log_e x + \frac{y}{x}, \quad \text{therefore} \quad \frac{dy}{dx} = \frac{y^2}{x - xy \log_e x}.$$

2. $y = a^{x^{x^{\&c.}}}$, therefore $\log_e y = x^{x^{x^{\&c.}}} \log_e a$; hence, putting in the previous formula $\dfrac{\log_e y}{\log_e a}$ for y, we get the required result.

3. $y = e^{x^{x^{\&c.}}}$ $\log_e y = x^{x^{x^{\&c.}}} \log_e e = x^{x^x}$; hence putting, in the formula $\dfrac{dy}{dx} = \dfrac{y^2}{x - xy \log_e x}$, $\log_e y$ for y, we get the required result.

10424. (W. J. BARTON, M.A.)—Show that, if the sum of two integers is a square, and the sum of their squares a square, (1) the sum of their cubes is the sum of two squares; (2) their product is always divisible by 12; (3) if they are both even, their product is divisible by 192.

Solution by R. F. DAVIS, M.A.; Prof. ZERR, M.A.; *and others.*

If P, Q be two integers such that $P^2 + Q^2$ is a square; then $P = x^2 - y^2$, $Q = 2xy$. If, in addition, $P + Q$ is a square, $x = m^2 + n^2$, $y = 2n\,(m - n)$. [Thus $40 + 9 = 7^2$, $1600 + 81 = 41^2$].

(1) Let $P + Q = R^2$ and $P^2 + Q^2 = S^2$; then

$$P^3 + Q^3 = (P + Q)(P^2 - PQ + Q^2) = \tfrac{1}{2}R^2\left\{S^2 + (P \sim Q)^2\right\}$$

$$= R^2\left\{\tfrac{1}{4}(S + P - Q)^2 + \tfrac{1}{4}(S - P + Q)^2\right\} = \text{sum of two squares};$$

for, since $S^2 = P^2 + Q^2$, and Q is always even, therefore S, P are both odd or both even.

(2) We here require to show that $xy(x^2-y^2)$ always $= M(6)$; x, y being any positive integers. Supposing x and y to be both odd, then $x \pm y$ are both even. Supposing x and y to be neither of them multiples of 3, they must therefore each be of the form $M(3) \pm 1$; hence one of the two $x \pm y$ is $M(3)$. Now $PQ = 2xy(x^2-y^2)$, and is therefore $M(12)$.

(3) Q is always even; and so is y. Hence P will be even, when x is even. Thus, when P, Q are both even, so also are x, y. Put $x = 2\xi$, $y = 2\eta$; then $PQ = 2xy(x^2-y^2) = 32\xi\eta(\xi^2-\eta^2) = 32 \cdot M(6) = M(192)$. The properties (2), (3) are therefore true with reference to any two integers the sum of whose squares is a square.

10491. (Professor MATHEWS.)—Prove that, (1) if $Pa + Qb + Rc + Sd = 0$, then the result of changing x, y, x', y' into $by - ay'$, $cx - ax'$, $dy - cy'$, $dx - bx'$, in the expression $f \equiv Pxy + Qxy' + Rx'y + Sx'y'$, is $(bc - ad)f$; and (2) that a similar theorem is true when

$$P : Q : R : S = d : -c : -b : a.$$

Solution by Professors ZERR, IGNACIO BEYENS; *and others.*

The result of direct substitution is

$$\{Pbc + d(Qb + Rc + Sd)\}\, xy - \{Qad + c(Pa + Rc + Sd)\}\, xy'$$
$$- \{Rad + b(Pa + Qb + Sd)\}\, x'y + \{Sbc + a(Pa + Qb + Rc)\}\, x'y'.$$

In the first case this becomes $(bc - ad)f$, and in the second $(ad - bc)f$.

10485. (Professor DÉCAMPS.)—Résoudre les équations

$$x^2 + y^2 = a, \quad z^2 + u^2 = b, \quad xy + zu = c, \quad xu + yz = d.$$

Solution by WALTER STOTT; Rev. J. L. KITCHIN; *and others.*

Put $xy - zu = A$; then, by the first three equations,

$$A = (a^2 - b^2)c + 2d\left\{(ab - d^2)\left[(a-b)^2 - 4(c^2 - d^2)\right]\right\}^{\frac{1}{2}} / \left\{(a-b)^2 + 4d^2\right\};$$

hence, substituting A in the fourth equation, we have

$$\left[(a + c + A)(b + c - A)\right]^{\frac{1}{2}} + \left[(a - c - A)(b - c + A)\right]^{\frac{1}{2}} = 2d,$$
$$x = \tfrac{1}{2}\left[(a + c + A)\right]^{\frac{1}{2}} + \left[(a - c - A)\right]^{\frac{1}{2}},$$
$$y = \tfrac{1}{2}\left[(a + c + A)\right]^{\frac{1}{2}} - \left[(a - c - A)\right]^{\frac{1}{2}},$$
$$u = \tfrac{1}{2}\left[(b + c - A)\right]^{\frac{1}{2}} + \left[(b - c + A)\right]^{\frac{1}{2}},$$
$$z = \tfrac{1}{2}\left[(b + c - A)\right]^{\frac{1}{2}} - \left[(b - c + A)\right]^{\frac{1}{2}}.$$

9162. (Professor BYOMAKESA CHAKRAVARTI, M.A.)—Show that the reciprocal polar of the evolute of the hyperbola $b^2x^2 - a^2y^2 = a^2b^2$, with regard to the circle described on the line joining the foci as diameter, is the curve $(a^2/x^2) - (b^2/y^2) = 1$, which, when the hyperbola is equilateral, becomes $r^2 \sin 2\theta + 4a^2 \cot 2\theta = 0$.

Solution by Rev. T. GALLIERS, M.A.; Prof. BHATTARYA; *and others.*

If $A = c^2/a$, $B = c^2/b$, $c^2 = a^2 + b^2$, the equation of the evolute of the hyperbola is $(x^{\frac{2}{3}}/A^{\frac{2}{3}}) - (y^{\frac{2}{3}}/B^{\frac{2}{3}}) = 1$ ..... (1).

If θ be the angle of inclination to the axis of x of a tangent to (1), the equation of the tangent will be

$$x \sin\theta - y \cos\theta = AB \sin\theta \cos\theta / (B^2 \cos^2\theta - A^2 \sin^2\theta)^{\frac{1}{2}} \text{ ..... (2)},$$

and if p = length of the perpendicular on (2) from the centre of the hyperbola, $p = c^2 \sin\theta \cos\theta / (a^2 \cos^2\theta - b^2 \sin^2\theta)^{\frac{1}{2}}$ ..... (3).

Let p be produced to a point Q (X, Y), and let $OQ = \rho$. Then, if Q be a point on the reciprocal polar of (1) with regard to the circle described on the line joining the foci as diameter,

$$p\rho = c^2 \text{ .....(4)};$$

also, $X = -\rho \sin\theta$, $Y = \rho \cos\theta$,

$$\therefore\ (a^2/X^2) - (b^2/Y^2) = (a^2 \cos^2\theta - b^2 \sin^2\theta)/\rho^2 \sin^2\theta \cos^2\theta,$$

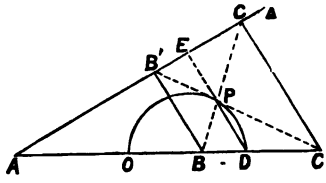
that is, by (3), (4), $(a^2/X^2) - (b^2/Y^2) = 1$ ..... (5).

If in (5) $b = a$, the equation becomes, when transformed to polar coordinates, $r^2 \sin 2\theta + 4a^2 \cot 2\theta = 0$.

10568. (Professor DE LONGCHAMPS.)—Soient A, B, C trois points en ligne droite. Par A, on fait passer une droite mobile Δ, et des points B, C on abaisse, sur Δ, des perpendiculaires BB′, CC′. Trouver le lieu décrit par le point de concours des diagonales du trapèze ainsi formé.

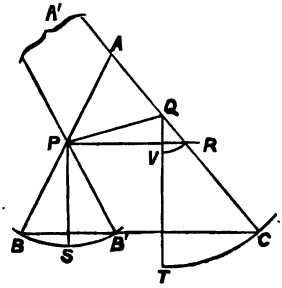
Solution by J. C. ST. CLAIR; Professor ANDERSON; *and others.*

Let P be the intersection of the diagonals BC′, CB′. Take D harmonic conjugate of A with respect to BC, and let DP meet B′C′ in E. Then, from the harmonic properties of the quadrilateral BB′C′C, DE is parallel to BB′, CC′, and is bisected in P. And, since E lies on a circle having AD for diameter, P, being the mid-point of the chord DE, lies on another circle of which the diameter DO is $\frac{1}{2}$AD.



4112. (C. H. Hinton, M.A.)—Given two points, one on each side of a triangle, the distances of each from the base-vertex of that side, the angle at which the line joining these points cuts the base, and the ratio of the sides: construct the triangle.

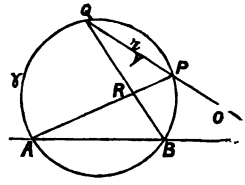
Solution by D. Biddle *and* Morgan Brierley.



Let P, Q be the given points, QPR = the given angle, PS, QT = the given distances, and PS : TV = the given ratio. Describe the arcs VR, TC, SB; through R draw QC, and draw CB parallel to PR. Then it is clear that we have two triangles fulfilling the conditions, viz., ABC and A'B'C; and this is always the case; but, when TV < S, the alternative sides pass through PQ, instead of P.

10590. (Professor Genese, M.A.)—A, B are two points of a circle, O any point in its plane; OPQ is any secant, and AP, BQ meet at R. The locus of R is a conic through A, B. Prove that, by properly choosing the circle and point O, the locus may be made to coincide with any given conic through A, B.

Solution by Professor Schoute.



When OPQ turns about O, the points P and Q form an involution on the circle γ through A and B. Therefore the pencils (AP) and (BQ) are in (1, 1) correspondence, and R describes a conic through A, B. By properly choosing the circle γ and the point O, the locus may be made to contain A, B and three points R, S, T of the plane chosen at random; this can be proved in the following manner. When γ describes the pencil of circles through A and B, the join r_γ of the points P and Q of γ on AR and BR describes a pencil of parallel lines; for the angle PQB is equal to PAB, and is therefore constant. This pencil r_γ is in (1, 1) correspondence with the pencil of circles. In the same manner the pencil of circles generates two pencils of parallel lines s_γ and t_γ by means of the lines AS and BS, AT and BT through the given points S and T. Now the pencils r_γ, s_γ, t_γ are mutually in (1, 1) correspondence, as this relation holds between the pencil of circles and each of them. And, as the line at infinity corresponds with itself in the three pencils, the loci of the point of intersection of the corresponding lines of each couple are three concurrent lines. Evidently the point common to these three lines will be the point O, which, in combination with the circle γ corresponding to the three lines r_γ, s_γ, t_γ through this point O, generates a conic passing through A, B, R, S, T.

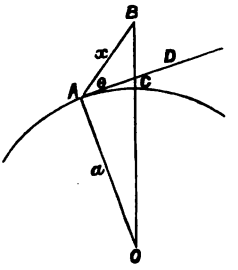
4193. (Professor Evans, M.A.)—At the station A, the apparent angular elevation of a meteor B, whose distance from the earth's surface is one-nth of the earth's radius, is θ. Supposing the earth to be a perfect sphere, find the exact distance from A to B.

Solution by Professor Asaph Hall, M.A.

Let AD be a tangent at A; then we have

$$AO = CO = a,$$
$$\angle BAO = 90° + \theta,$$
$$BC = a/n,$$
$$BO = \frac{(n+1)\,a}{n};$$

hence in the plane triangle BAO two sides and an angle given, and the distance follows at once.

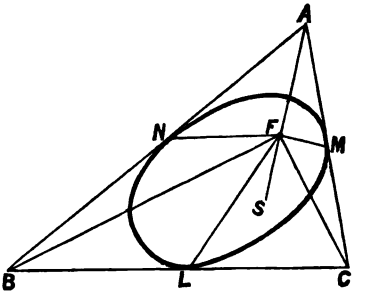


10562. (Professor Morley.)—Let there be three parallel rectilinear vortices in an infinite mass of fluid. Let a cross section meet the vortices at ABC. Prove that (1) the motion of any vortex A is at right angles to AK, where K is the symmedian point; and (2) the particles at foci of the maximum ellipse inscribed in ABC are instantaneously at rest.

Solution by Professor Anderson, M.A.

(1) If the vortices be of equal strengths m, the velocity of the point A due to the vortex B is perpendicular to AB and equal to m/c. In like manner, the velocity of A due to the vortex C is perpendicular to AC and equal to m/b. The resultant velocity of the point A is easily seen, from this, to be perpendicular to the symmedian through A.

(2) Draw the maximum ellipse of the triangle ABC touching at L, M, N the mid-points of the sides, and let F be one of its foci. Join FA, FB, FC, FL, FM, FN, and produce AF to S.



$$\angle SFC = \pi - AFC = \pi - (NFA + LFC)$$
$$= \pi - \tfrac{1}{2}(NFM + LFM) = \tfrac{1}{2}(2\pi - NFM - LFM) = \tfrac{1}{2}NFL = BFL.$$

Hence, since FL is the median of the triangle FBC through F, FS is the symmedian through F of the same triangle. Now the velocity of the

point F due to the vortices B and C is by (1) perpendicular to FS, and its velocity due to A is perpendicular to the same line. Its resultant velocity is therefore perpendicular to FA. But, in the same way, it may be proved that it is perpendicular to FB and FC. Hence F is at rest, and in the same way the other focus may be shown to be at rest.

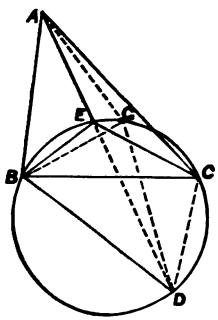
10210. (E. M. Langley, M.A.)—Find a point E within the triangle ABC such that $l \cdot AE + m \cdot BE + n \cdot CE$ may be a minimum; l, m, n being any three constants such that two are together greater than the third.

Solution by R. Chartres, M.A.; J. J. Barniville; *and others.*

On BC draw a triangle BCD such that

$$BC : CD : DB = l : m : n;$$

then the point required will be the intersection E of AD with the circumcircle of BCD, if E is within the triangle ABC.



For $BE \cdot CD + CE \cdot BD = ED \cdot BC$,

therefore $m \cdot BE + n \cdot CE = l \cdot ED$;

therefore $l \cdot AE + m \cdot BE + n \cdot CE = l \cdot AD$.

But, if G is any other point on the arc BEC,

$$m \cdot BG + n \cdot CG = l \cdot GD;$$

$\therefore\ l \cdot AG + m \cdot BG + n \cdot CG = l \cdot AG + l \cdot GD;$

$\therefore\ l \cdot AG + m \cdot BG + n \cdot CG > l \cdot AD.$

And if P be any point within the triangle ABC, but not on the circle,

$$BP \cdot CD + CP \cdot BD > PD \cdot BC, \quad \therefore\ m \cdot BP + n \cdot CP > l \cdot PD;$$

therefore $l \cdot AP + m \cdot BP + n \cdot CP > l \cdot AP + l \cdot PD;$

therefore $l \cdot AP + m \cdot BP + n \cdot CP > l \cdot AD.$

If l, m, n are proportional to a, b, c, E is the orthocentre of ABC.

If l, m, n are proportional to c, a, b, or b, c, a, E is one of the Brocard-points of ABC, and the construction for E is equivalent to that of Mr. R. F. Davis for the Brocard-points [Vol. xlvii., Appendix II.].

The triangle formed by perpendiculars to AE, BE, CE through A, B, C, is the maximum triangle through A, B, C, with its sides proportional to l, m, n.

[Quest. 10562 has an intimate connexion with the following Question proposed by Mr. Morgan Jenkins in the *Educational Times* for August, 1884:—

"If on the three sides of a triangle, ABC, there be described any three triangles, BDC, CEA, AFB, either all externally or all internally having their angles in the same order of rotation, and the angles which are contiguous to the same corner of ABC equal to each other, prove that AD,

BE, and CF meet in a point O, which is also the common point of intersection of the circumcircles of BDC, CEA, AFB" (Vol. XLIII., pp. 88–91).

Mr. CHARTRES remarks that this Question and mode of proof were suggested by a paper of his on FERMAT's problem (to find EA + EB + EC a minimum), published in *Nature* for Feb. 2, 1888.]

10638. (J. GRIFFITHS, M.A.)—If a, b, c, k be all real quantities, and k not greater than unity, prove that $(a+b-2kc)^2 > 4(1-k^2)(ab-c^2)$.

Solution by R. H. W. WHAPHAM, B.A.; M. BRIERLEY; *and others.*

We have to prove that $(a-b)^2+4(c-ak)(c-bk)>0$, which is evident if the factors $c-ak$, $c-bk$ are both positive or both negative; but suppose only one of them $c-ak$ (say) is negative ($=-\delta^2$), and $c-bk=\epsilon^2$; then $a-b=(\delta^2+\epsilon^2)/k$; thus we have to show that $[(\delta^2+\epsilon^2)/k]^2-4\delta^2\epsilon^2>0$, which is clear if $k=1$, and, *à priori*, when $k<1$.

10610. (D. BIDDLE.)—Give a general method of solving Question 9957, already solved in a particular case in Vol. LI., pp. 67, 68.

Solution by J. D. HAMILTON DICKSON, M.A.

The question is to find that function of x which, when x is replaced by 1, 2, 3, 4, yields respectively 0, $\frac{3}{5}$, $\frac{17}{18}$, $\frac{641}{500}$.

This is a mere application of LAGRANGE's (or any other) interpolation formula, or of the well-known formula in Finite Differences which gives u_x in terms of its successive differences. It is obvious, however, that an infinite number of functions fulfilling the required conditions may be found, and that therefore some one at least will be simpler than the others. It follows also that, if U, V be two such functions, then U and V must, generally speaking, differ by some multiple of

$$(x-1)(x-2)(x-3)(x-4) \equiv \phi(x).$$

Now, whatever the simplest function may be, calling it $f(x)$, let any other function fulfilling the conditions of the question be called $F(x)$, and let X, Y, Z, W be *any* functions of x; then a little consideration will show that we may write, in general, (1) below; or, neglecting the term in W, as no one would select a more complicated function than was necessary, we obtain (2), a form not unlikely to occur; and another form, even more likely to occur, would be (3):—

$$F(x) = W \cdot \phi(x) + \frac{X\{Y\phi(x)+f(x)\}}{X\{Z\phi(x)+1\}} \quad \ldots\ldots\ldots\ldots\ldots\ldots (1),$$

$$F(x) = \frac{X\{Y\phi(x)+f(x)\}}{X\{Z\phi(x)+1\}}, \quad F(x) = Y\phi(x)+f(x) \quad \ldots\ldots\ldots (2, 3).$$

The values of $F(x)$ in (2) and (3) would *appear* fully multiplied out, so that the $f(x)$ would not be explicitly visible, as shown. Also, (2) and (3) suggest steps for arriving (if possible) at a simpler function than $F(x)$.

Both LAGRANGE's formula, and the Finite Difference formula show that the function, which represents n given values for n separate values of x, need not be of more than $n-1$ dimensions in x. Thus Mr. KNOWLES's formula $(64800x^5-\ldots)$ is at least *two* degrees higher than is necessary. Again, Mr. CHRISTIE's formula (which may be obtained by either of the above methods) is the quotient of *two* functions of the simplest kind, and ought therefore also to be capable of simplification.

I have been unable to find a simpler function than the following one given on p. 68, $f(x) = (-34x^3+279x^2+601x-846)/5400$ ..... (4).

Mr. KNOWLES' function (p. 67) is obviously a case of the formula (3), and is equal to $f(x)+\left(x+\frac{47}{7200}\right).\phi(x)$ .....(5).

Mr. CHRISTIE's formula is incorrect for the value $x=1$, as it then takes the indeterminate form $\frac{0}{0}$, but it gives the other three values correctly. Recalculating his denominator from the denominators of the numbers $\frac{0}{0}$, $\frac{5}{6}$, $\frac{17}{12}$, $\frac{641}{600}$ (the first of these numbers shows how the error arose), I find that his function takes the form (slightly different forms would be got by giving the first of these numbers any other denominator),

$$(596x^3-3537x^2+6451x-3510)/27\,(30x^3-177x^2+321x-172),$$

which must differ from $f(x)$ by a multiple of $\phi(x)$. On making the calculation, it is found to be

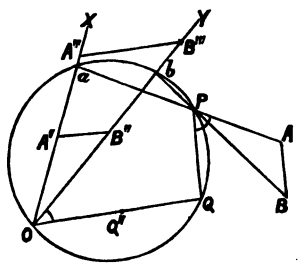
$$= f(x)+\frac{1020x^3-4188x-35313}{5400\,(30x^3-177x^2+321x-172)}.\phi(x)\ \ldots\ldots\ldots\ldots(6).$$

Formula No. (4) appears, therefore, to be the simplest (α) in dimensions of x, (β) in form, and (γ) in the magnitudes of its coefficients. [The PROPOSER states that the successive amounts in the Question are given by $\Sigma\{2(n-1)(n+1)^2\}$].

10588. (Professor NEUBERG.)—La droite AB est fixe ; la droite A′B′ se meut parallélement à elle-même entre deux droites fixes OX, OY. Trouver le lieu du centre de similitude de deux figures semblables construites sur AB, A′B′.

Solution by Professor SCHOUTE.

Soit A″B″ une des deux positions de la droite mobile A′B′ dans lesquelles la longueur de cette droite variable est égale à AB. Soit P le point de la figure (AB) qui correspond au point O de la figure (A″B″). Soit PQ une droite quelconque par P de la première figure et OQ la droite correspondante de la seconde, ce qui exige l'égalité des angles BPQ et B″OQ. Soit Q″ le point de la seconde figure qui correspond au point Q de la première (OQ″ = PQ).



Cela posé, on trouve que Q est le point double de la figure (AB) et de la

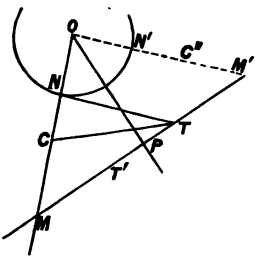
figure (A‴B‴) déterminée par la position A‴B‴ de la droite mobile A′B′ qui satisfait à la condition A‴B‴/A″B″ = OQ/OQ″, eu égard au signe.

Le lieu en question est donc la circonférence de cercle, qui passe par O, P et les points d'intersection a et b des couples de droites OX et AP, OY et BP.

[Construisons le triangle ABP semblable à A″B″O ; soit A l'intersection de AP avec OX. Le centre de similitude de deux figures semblables construites sur AB et A″B″, ou ce qui est la même chose sur AP et A″O, est à l'intersection des circonférences aOP, aA″A. La circonférence aOP est donc le lieu demandé.]

10579. (Morgan Brierley.)—A variable circle touches another given in magnitude and position, and also a line given in position; determine the magnitude and position of the variable circle when the line joining the centres makes a given angle with the given line.

Solution by J. C. St. Clair; Prof. Aiyar; *and others.*



Let O be the centre of the fixed circle, and OP the perpendicular on the given line. On either side of OP make ∠ POM = complement of the given angle, and draw the tangent at N, the point where OM cuts the circle. Let this tangent meet the given line in T, and let the bisector of ∠ NTP meet OM in C. Then, since ∠ CMP is equal to the given angle, and C is equidistant from the circle and line, C is the centre, and CN the radius, of the required circle.

It is obvious that the two circles satisfying the conditions are reflexions of each other with respect to OP.

9406. (W. J. Barton, M.A.)—Prove that, if R = 2r, the triangle is equilateral.

Solution by Professor Lampe, LL.D.

Professor Emmerich (in Vol. xlix., p. 70) bases his proof on the properties of the nine-point circle. An easy proof results from a few formulæ of trigonometry. Transforming

$$r = 4R \cdot \sin\tfrac{1}{2}\alpha \sin\tfrac{1}{2}\beta \sin\tfrac{1}{2}\gamma = 2R \sin\tfrac{1}{2}\alpha \left[\cos\tfrac{1}{2}(\beta-\gamma) - \cos\tfrac{1}{2}(\beta+\gamma)\right],$$

we get

$$\cos\tfrac{1}{2}(\beta-\gamma) = \frac{r}{2R \sin\tfrac{1}{2}\alpha} + \sin\tfrac{1}{2}\alpha ;$$

and as a cosine is $\leqq 1$ and $R = 2r$, we have

$$\frac{1}{4\sin\frac{1}{2}\alpha} + \sin\tfrac{1}{2}\alpha \leqq 1, \text{ or } \sin^2\tfrac{1}{2}\alpha - \sin\tfrac{1}{2}\alpha + \tfrac{1}{4} \leqq 0;\ i.e.,\ (\sin\tfrac{1}{2}\alpha - \tfrac{1}{2})^2 \leqq 0.$$

But a square is always > 0, unless it be $= 0$; therefore $\sin\frac{1}{2}\alpha = \frac{1}{2}$, $\alpha = 60°$, and $\cos\frac{1}{2}(\beta-\gamma) = 1$; $\beta = \gamma = 60°$. It would not be difficult to make a geometrical interpretation of this reasoning.

Another way was published by me five years ago in a collection of geometrical problems whose solutions lead to equations of higher degrees. Being given the sum of sides $2s$, the radii R and r of a triangle, we get the sides as the roots of $x^3 - 2sx^2 + (s^2 + 4r\rho + \rho^2)x^2 - 4r\rho s = 0$, whose discriminant is

$$\Delta = \left[s^2 - 2r^2 - 10r\rho + \rho^2 + 2r^{\frac{1}{2}}(r-2\rho)^{\frac{1}{2}}\right]\left[s^2 - 2r^2 - 10r\rho + \rho^2 - 2r^{\frac{1}{2}}(r-2\rho)^{\frac{1}{2}}\right].$$

Thence we conclude $r \geqq 2\rho$. If $r = 2\rho$, Δ becomes a square, and must therefore be $= 0$, if the cubic equation shall have three real roots, or $s^2 = 27\rho^2$, $s = 3\rho\sqrt{3}$. But in this case the three roots are $a = b = c = 2\rho\sqrt{3}$.

10606. (R. Tucker, M.A.)—P $(am_1^2, 2am_1)$, Q (m_2), R (m_3) are three co-normal points on a parabola: show that the rectangular coordinates of the "S"-point of PQR are given by

$$\Sigma(m)^2 \cdot Y = -6am_1m_2m_3,$$

$$2\Sigma(m^2)\cdot\left[\Sigma(m^2)+24\right]X = a\left[\left\{\Sigma(m^2)\right\}^2\cdot\left\{\Sigma(m^2)+8\right\} - 36m_1^2m_2^2m_3^2\right].$$

Solution by the Proposer.

In this case $\Sigma(m) = 0$, and therefore $m_1^2 - m_2m_3 = \Sigma(m^2)/2$. I write

$$\mu \equiv m_1m_2m_3, \quad p \equiv \Sigma(m^2)+8, \quad q \equiv m_1^2+8, \quad r \equiv \Sigma(m^2)+24.$$

The equations to the circle PQR, and the tangents to it at Q, R, are

$$x^2 + y^2 - \tfrac{1}{2}apx - \tfrac{1}{2}a\mu y = 0 \quad \ldots\ldots(1),$$

$$X(4m_2^2 - p) + Y(8m_2 - \mu) = am_2(pm_2 + 2\mu) \quad \ldots\ldots(2),$$

$$X(4m_3^2 - p) + Y(8m_3 - \mu) = am_3(pm_3 + 2\mu) \quad \ldots\ldots(3);$$

whence (2) − (3), and m_3^2 (2) − m_2^2 (3) give

$$4m_1X - 8Y = a(pm_1 - 2\mu), \quad pm_1X + qm_2m_3Y = 2am_2m_3\mu \ldots\ldots(4,\ 5).$$

From (4) and (5) we get point of intersection of tangents

$$4X\left[m_2m_3q + 2p\right] = am_2m_3\left[pq - 2m_1\mu\right] \quad \ldots\ldots\ldots\ldots\ (6),$$

$$4Y\left[m_2m_3q + 2p\right] = a\left[2\mu(4m_2m_3 + p) - m_1p^2\right] \quad \ldots\ldots\ (7).$$

Hence "S"-line through P [*i.e.*, join of P and (6, 7)], is

$$r\left[2\mu + pm_1\right]X + \left\{2m_1\mu\left[\Sigma(m^2) - 3m_1^2 - 12\right] - 4\mu\Sigma(m^2)\right\}Y$$
$$= am_1\left[m_1^2pr - 12\mu^2 - 8p\Sigma(m^2)\right] \ldots\ldots\ldots\ldots(8).$$

Adding (8) and two analogous equations,

$$2\mu rX - 2\left[2p\Sigma(m^2) + 3\mu^2\right]Y = a\mu pr \quad \ldots\ldots\ldots\ldots\ (9)$$

also, from (8) and the "S"-line through Q, we get

$$pr\,X - \mu r\,Y = a\left[p^2\Sigma(m^2) - 24\mu^2\right]/2 \quad \ldots\ldots\ldots\ldots \quad (10).$$

From (9) and (10) we get, removing the factor $\mu^2 + p^2$,

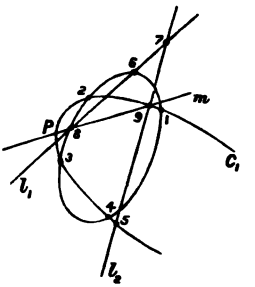
$$\Sigma(m^2)\,Y = -6a\mu,$$

and, by substitution, $2r\Sigma(m^2)\,X = a\left\{p\left[\Sigma(m^2)\right]^2 - 36\mu^2\right\}$.

Imposing conditions on the position of the co-normal point, we can get the locus, in several cases, of the corresponding "S"-point.

2874. (The late T. Cotterill, M.A.)—Seven points on a cubic locus have an opposite point on the curve: *i.e.*, a variable cubic through seven given points cuts a fixed cubic through the same points, in two other points collinear with a point on the fixed cubic. Construct for the opposite point when the fixed cubic breaks up into a conic through five points and a line through two.

Solution by Professor Schoute.



The fixed cubic breaks up into the conic C_1 determined by the points 1, 2, 3, 4, 5, and the line l_1 joining the points 6, 7. The construction consists of three equal parts.

1. By means of Pascal's theorem we seek the second point 8 common to C_1 and line l_2 joining 5, 7.

2. In the same manner we determine the second point 9 common to the conic C_2 through 1, 2, 3, 4, 6, and l_1.

3. Finally, we construct in the same way the second point P common to C_1 and the join m of 8, 9.

This point P is the opposite point in question; it is determined by linear construction without tracing the conics, the diagram only serving for illustration. [See also Vol. LVII., p. 155.]

10594. (Professor Wolstenholme, M.A., Sc.D.)—On the parabola $y^2 = 4ax$ is taken the point P $(am^2, 2am)$, and the coordinates of a point Q are $[a(8+m^2), -\frac{8}{3}am]$; prove that the circle described with centre Q and radius QP will meet the parabola again in three points which are corners of an equilateral triangle, and that QP is always normal to the parabola $y^2 = 36a(x + 12a)$.

Solution by Professor ANDERSON; J. D. WILLIAMS; *and others.*

Let P, R, S, T be the four points of intersection of the circle and parabola; then the triangle RST will be equilateral if the point Q be its centroid. The equation of the circle is

$$\left[x-a\left(8+m^2\right)\right]^2+\left[y+\tfrac{3}{2}am\right]^2-64\left(a^2+\tfrac{1}{16}a^2m^2\right)=0.$$

Substituting for x, $a\mu^2$, and for y, $2a\mu$, the resulting equation in μ will give the points of intersection; and, since P is one of these points, the equation is divisible by $\mu-m$. Performing the division, we obtain the cubic

$$\mu^3+m\mu^2-\mu\left(m^2+12\right)-\left(m^3+\tfrac{3}{2}^2m\right)=0.$$

Let the roots of this equation be μ_1, μ_2, μ_3, and the coordinates of the points R, S, T, x_1y_1, x_2y_2, x_3y_3. Hence

$$\tfrac{1}{3}\left(y_1+y_2+y_3\right)=\tfrac{1}{3}\left\{2a\left(\mu_1+\mu_2+\mu_3\right)\right\}=-\tfrac{1}{2}am;$$

$$\tfrac{1}{3}\left(x_1+x_2+x_3\right)=\tfrac{1}{3}a\left(\mu_1^2+\mu_2^2+\mu_3^2\right)=\tfrac{1}{3}a\left(3m^2+24\right)=a\left(m^2+8\right);$$

and these are the coordinates of the point Q.

Again, the equation of the line PQ is $3y+mx-6am-am^3=0$. The envelope of this line, when m is made to vary, is $243ay^2=4\left(x-6a\right)^3$. But the evolute of the parabola $y^2=4c\left(x+b\right)$ is easily shown to be

$$27cy^2=4\left(x+b-2c\right)^3.$$

Identifying these equations, we have $c=9a$, $b=12a$. The line PQ therefore touches the evolute of, or is normal to, the parabola

$$y^2=36a\left(x+12a\right).$$

10464. (Professor ZERR, M.A.)—Solve the equations

$$x+y^2+z^3=21,\quad x^2+y^3+z=45,\quad x^3+y+z^2=71.$$

Solution by J. BOLAND, M.A.; Prof. MUKHOPÂDHYÂY; *and others.*

The equations may be reduced, in various ways, to a quadratic, from which we readily find one set of values to be $x=4$, $y=3$, $z=2$.

10563. (Professor CATALAN.)—Résoudre, complètement, l'équation

$$\left(x^3-3qx+p^3-3pq\right)^2-4\left(px+q\right)^3=0.$$

Solution by Professors SIRCOM, BEYENS; *and others.*

This is the condition that $y^3-3\left(px+q\right)y+x^3-3px+p^3-3pq=0$ may have equal roots. These roots are given by

$$y+x+p=0,\quad y^2-\left(p+x\right)y+x^2-px+p^2-3q=0;$$

whence, for equal roots, $2(p+x)^2+x^2-px+p^2-3q=0$;

whence $x^2-2px+p^2-4q=0$ or $x^2+px+p^2-q=0$.

The roots of these equations are the roots of the given equation, those of the second being reckoned twice.

10616. (S. Tebay, B.A.)—An urn contains florins and half-crowns amounting to £1. 13*s*. If a person draw three coins at random, show that the value of his expectation is 6*s*. $7\frac{61}{140}d$.

Solution by W. J. Greenstreet, M.A.; Fabian Rosenberg; *and others.*

Let x, y = numbers of florins and half-crowns; then, in sixpences we have $4x+5y=66$, whereof the solution is $x=14-5t$, $y=2(2t+1)$; hence the only suppositions in florins and half-crowns are (14, 2), (9, 6), (4, 10); and these hypotheses are equally likely; thus the expectations, in shillings, are respectively $\frac{1}{16}(28+5)$, $\frac{1}{16}(18+15)$, $\frac{1}{16}(8+25)$; and the total expectation $=33(\frac{1}{16}+\frac{1}{16}+\frac{1}{16})$ shillings, which gives the stated result.

4132. (Matthew Collins, LL.D.)—The equation of a curve of the *n*th order being $P+Q+R+S+\&c.=0$, where $P=ax^{n-2}y^2+a'x^{n-3}y^3+\&c.$, $Q=bx^{n-1}+b'x^{n-2}y+\&c.$, $R=cx^{n-2}+c'x^{n-3}y+\&c.$; show that Euler has fallen into error in giving $ay^2+bx=0$ as the equation of the asymptotic parabola, and find the *correct* equation.

Solution by Professor Sebastian Sircom, M.A.

Making the equation to the curve homogeneous by introducing z, where $z=0$ is the line at infinity, we must have $z^2=0$ if the equation of the parabola $ay^2+(bx+qy)z=0$ be substituted. This gives

$$0=-(bx+qy)x^{n-2}z-\frac{a_1}{a}(bx+qy)x^{n-3}yz+bx^{n-1}z+b_1x^{n-2}yz+\ldots,$$

whence $$b_1-q-\frac{a_1}{a}b=0,\quad q=\frac{ab_1-a_1b}{a},$$

and the asymptotic parabola is $a^2y^2+abx+(ab_1-a_1b)y=0$.

10249. (Professor Lemoine.) — Construire un tétraèdre équifacial SABC, connaissant la droite sur laquelle est l'arête SA, le plan mené par SA parallèlement à BC, un point de BC, enfin un triangle LMN semblable à ABC.

Solution by Professor IGNACIO BEYENS.

Désignons par a le plan donné, par P le point donné sur BC. Si l'on construit sur LMN un tétraèdre équifacial OLMN il sera semblable à SABC, et les angles (SA, BC), (OL, MN) seront égaux. Donc on peut obtenir la droite indéfinie BC en construisant, dans le plan mené par P parallèlement à (a) une droite faisant avec une parallèle à SA par P un angle égal à l'angle (OL, MN). Soient maintenant DE, QR les plus courtes distances des couples SA, BC, OL, MN. Les proportions

$$DE : QR = DS : QO = DA : QZ = EB : RM = EC : RN$$

serviront à déterminer les segments inconnus DS, DA, EB, EC.

2632. (The EDITOR.)—Prove that

$$1 . 2 . 3 \ldots n \equiv n! < 2^{2n(n-1)}.$$

Solution by Professor LAMPE, LL.D.

Writing $1 = 1$, $2 = 2$, $3 < 2^2$, $4 = 2^2$, $5 < 2^3$, $6 < 2^3$, $7 < 2^3$, $8 = 2^3$, $9 < 2^4$, $10 < 2^4$, ... , $16 = 2^4$, ... , and multiplying, we get another limit for $m = 2^n$,

$$1 . 2 . 3 \ldots, \; m < 2^{1+2.2+3.2^2+4.2^3+\ldots+n.2^{n-1}};$$

the exponent being $= (n-1)\,2^n + 1 = (n-1)\,m + 1$, we have

$$1 . 2 . 3 \ldots m < 2^{nm-m+1} \text{ or } < 2 . (2^n/2)^m, \; i.e., \; 2 . (\tfrac{1}{2}m)^m.$$

For instance, $1 . 2 . 3 \ldots 16 < 2 . 8^{16}$, *i.e.*, $< 2^{49}$;

while conforming to the above formula it is only $< 2^{120}$.

As a corollary we conclude a limit for $(m+a)!$, a being $< m$;

$$(m+1) < 2m, \; (m+2) < 2m, \; (m+3) < 2m, \ldots, \; (m+a) < 2m,$$

$$(m+1)(m+2) \ldots (m+a) < 2^a . m^a,$$

or $$(m+a)! < 2\,(\tfrac{1}{2}m)^m . 2^a . m^a, \; i.e., \; < 2^{a+1} . (\tfrac{1}{2}m)^m . m^a.$$

For instance, $19! < 2^4 . 8^{16} . 16^3$, *i.e.*, $< 2^{64}$.

[Another proof of the theorem is given on p. 85 of Vol. LII.]

10522. (Professor NEUBERG.)—Soient un triangle équilatéral ABC et une circonférence concentrique ; tous les triangles dont les sommets sont les projections, sur les côtés de ABC, d'un point mobile sur cette circonférence, ont même angle de Brocard.

Solution by Professors ANDERSON, SARKAR, *and others.*

Let us take a more general theorem, and find the locus of a point P such that the Brocard-angle of the pedal triangle of this point with

respect to a given triangle ABC shall be constant. If α, β, γ be the angles of the pedal triangle, and ω the Brocard-angle, we must have

$$\cot\alpha + \cot\beta + \cot\gamma = \cot\omega = \text{constant}.$$

Let x, y, z be the trilinear coordinates of the point P with respect to the triangle ABC; then it is easily shown that

$$\cot\alpha = (-x^2 + yz\cos A - xz\cos B - xy\cos C)/(yz\sin A + xz\sin B + xy\sin C),$$

and similar equations hold for $\cot\beta$ and $\cot\gamma$. Hence, by addition,

$$x^2 + y^2 + z^2 + yz\cos A + xz\cos B + xy\cos C + \cot\omega\,(yz\sin A + xz\sin B + xy\sin C) = 0,$$

denoting a system of circles. If $\omega = 0$, we get

$$yz\sin A + xz\sin B + xy\sin C = 0,$$

the circumcircle of the triangle. If $A = B = C = \frac{1}{3}\pi$, the equation always represents a circle concentric with the incircle of the triangle.

10514. (Professor Genese, M.A. Suggested by Question 10497.)—A, B are fixed points of intersection of two conics; X, Y, the points of contact of a common tangent; and XY meets AB at T. Prove that the expression $XY\,(TA\,.\,TB)^{\frac{1}{2}}/(TX\,.\,TY\sin\theta)$, where θ is the angle between XY and AB, is independent of the choice of tangent.

Note by Professor Curtis, M.A.

In this Question, the expression $XY\,(TA\,.\,TB)^{\frac{1}{2}}/TX\,.\,TY\sin\theta$ is said to be independent of the choice of tangent. That can hardly be; for, if the two conics touch at a point X, at that point we have $XY = 0$, and therefore the expression $= 0$; but it does not follow that for any other common tangent the expression $= 0$.

10506. (Professor Wolstenholme.)—Corresponding to each point P of a given ellipse (or hyperbola) whose foci are S′, S′, is described another ellipse (or hyperbola) having a focus at S′, a vertex at P, and its major axis equal to the major axis of the given ellipse; prove that the only real common point of this conic and the given one are P, Q, where PQ is the chord normal at P.

Solution by Professors Anderson, Mukhopâdhyây, *and others*.

Let S, S′ be the foci of the ellipse; the normal at P intersect the ellipse again in N; and with radius NS′ draw a circle cutting PS in S″. Then $NS'' = NS'$, and, since $\angle SPN = \angle S'PN$, $PS' = PS''$. Hence

$$NS + NS'' = NS + NS' = PS + PS' = PS + PS''.$$

Hence an ellipse whose foci are S, S″, and vertex P, will pass through N, and its major axis will be equal to that of the original ellipse.

P and N are the only real points of intersection. For suppose Q to be another such point on the original ellipse; then $QS + QS'' = QS + QS'$, and therefore $QS'' = QS'$, from which it follows that Q must be on PN, which can intersect the ellipse in only two points. In like manner, P and N are the only real points of intersection of the given ellipse and another with focus S', vertex P, and major axis equal to that of the former. A similar proof applies to the hyperbola.

10469. (Professor Lemoine.)—Soient I, I_a, I_b, I_c les centres des cercles tangents aux trois côtés du triangle ABC, et soient T, T_a, T_b, T_c les points de concours des droites joignant A, B, C aux points de contact des côtés opposés avec l'un de ces cercles. Démontrer que les droites IT, I_aT_a, I_bT_b, I_cT_c concourent en un même point T, et trouver les coordonnées de ce point.

Solution by Professors Anderson, Chakravarti, *and others.*

Take the triangle ABC as triangle of reference. The trilinear coordinates of the eight points in the question are proportional to the quantities set forth in the following scheme:—

I,	1,	1,	1	T,	$\sec^2\frac{1}{2}A$,	$\sec^2\frac{1}{2}B$,	$\sec^2\frac{1}{2}C$,
I_a,	-1,	1,	1	T_a,	$-\sec^2\frac{1}{2}A$,	$\operatorname{cosec}^2\frac{1}{2}B$,	$\operatorname{cosec}^2\frac{1}{2}C$,
I_b,	1,	-1,	1	T_b,	$\operatorname{cosec}^2\frac{1}{2}A$,	$-\sec^2\frac{1}{2}B$,	$\operatorname{cosec}^2\frac{1}{2}C$,
I_c,	1,	1,	-1	T_c,	$\operatorname{cosec}^2\frac{1}{2}A$,	$\operatorname{cosec}^2\frac{1}{2}B$,	$-\sec^2\frac{1}{2}C$.

The equations of the lines IT, I_aT_a, I_bT_b, I_cT_c are—

(1) $(\sec^2\frac{1}{2}C - \sec^2\frac{1}{2}B)\,x + (\sec^2\frac{1}{2}A - \sec^2\frac{1}{2}C)\,y + (\sec^2\frac{1}{2}B - \sec^2\frac{1}{2}A)\,z = 0,$

(2) $(\operatorname{cosec}^2\frac{1}{2}C - \operatorname{cosec}^2\frac{1}{2}B)\,x + (-\sec^2\frac{1}{2}A + \operatorname{cosec}^2\frac{1}{2}C)\,y + (-\operatorname{cosec}^2\frac{1}{2}B + \sec^2\frac{1}{2}A)\,z = 0,$

(3) $(-\operatorname{cosec}^2\frac{1}{2}C + \sec^2\frac{1}{2}B)\,x + (\operatorname{cosec}^2\frac{1}{2}A - \operatorname{cosec}^2\frac{1}{2}C)\,y + (-\sec^2\frac{1}{2}B + \operatorname{cosec}^2\frac{1}{2}A)\,z = 0,$

(4) $(-\sec^2\frac{1}{2}C + \operatorname{cosec}^2\frac{1}{2}B)\,x + (-\operatorname{cosec}^2\frac{1}{2}A + \sec^2\frac{1}{2}C)\,y + (\operatorname{cosec}^2\frac{1}{2}B - \operatorname{cosec}^2\frac{1}{2}A)\,z = 0.$

The determinant formed by the coefficients of (2), (3), and (4) vanishes on inspection, and therefore the lines I_aT_a, I_bT_b, I_cT_c meet in a point. Also, the four equations, when added together, vanish identically, showing that IT passes through this point. The coordinates of the point are proportional to the minors of any row of the above determinant.

It may be of interest to add that, if parallels to the opposite sides be drawn through the angular points A, B, C forming a new triangle A'B'C', and if G be the centre of gravity of the original triangle, the straight lines GT, $A'T_a$, $B'T_b$, $C'T_c$ meet in a point whose areal coordinates with reference to ABC are, if s = semi-sum of sides, proportional to

$$s^2(c - a + b) - 2bcs + abc, \quad s^2(a - b + c) - 2acs + abc,$$

and

$$s^2(b + a - c) - 2abs + abc.$$

10254. (L. W. Robinson.)—If $(1+x)^n = 1 + c_1x + c_2x^2 + \ldots$, prove that, according as n is even or odd,

$$\Delta \equiv \begin{vmatrix} 1, & c_1, & c_2 \ldots & c_n \\ c_1, & c_2, & c_3 \ldots & \\ \ldots & \ldots & \ldots & \\ c_n, & c_{n-1} & \ldots & \end{vmatrix} = 0 \quad \text{or} \quad 2^n.$$

Solution by Professor Mati Lul Mallik, M.A.

We have $\Delta = (1 + c_1 + c_2 \ldots c_n)(1 + \theta c_1 + \theta^2 c_2 \ldots) \ldots (1 + \theta^n c_1 + \theta c_2 + \ldots)$, where 1, θ_1, θ_1, θ_3, &c. are the roots of $x^{n+1} - 1 = 0$ .....(a)

(see Ex. 27, p. 234, Burnside and Panton's *Theory of Equations*).

Substituting for x in the given identity, respectively, 1, θ_1, θ^2, ... θ^n, we get, by virtue of the equation (a),

$$(1 + c_1 + c_2 \ldots c_n)(1 + \theta c_1 + \theta^2 c_2 \ldots) \ldots (1 + \theta^n c_1 + \theta c_2 \ldots),$$

equal to $(1 \mp 1)^n$, according as n is even or odd, or the determinant equal to 0 or 2^n.

10357. (E. M. Langley, M.A.)—If the centre of a circle be the centroid of four points on its circumference, prove that (1) they are vertices of a rectangle; and hence (2) if

$$\sin\alpha + \sin\beta + \sin\gamma + \sin\delta = 0 = \cos\alpha + \cos\beta + \cos\gamma + \cos\delta,$$

then $\sin(2n+1)\alpha + \sin(2n+1)\beta + \sin(2n+1)\gamma + \sin(2n+1)\delta = 0$,

and $\cos(2n+1)\alpha + \cos(2n+1)\beta + \cos(2n+1)\gamma + \cos(2n+1)\delta = 0$;

also (3) enunciate and prove a corresponding theorem for three angles.

Solution by Professors Anderson, Evans, *and others.*

Let A, B, C, D be the four points, E the middle point of AB, and F the middle point of CD. If O be the centre of the circle, the forces represented by OA, OB, OC, OD are in equilibrium. Hence OE is equal to and in the same straight line with OF, the former being also perpendicular to AB, and the latter to CD. It follows at once that the figure ABCD is a rectangle.

Again, the positions of A, B, C, D may be given by the angles α, β, γ, δ, measured from a fixed line through the centre of the circle, and, if the equations $\Sigma \sin\alpha = 0$, $\Sigma \cos\alpha = 0$ hold, the centre of the circle is the centroid of the points. Hence, since the points form a rectangle,

$$\gamma - \alpha = \pi, \quad \delta - \beta = \pi;$$

therefore $(2n+1)\gamma = (2n+1)(\pi + \alpha)$, $(2n+1)\delta = (2n+1)(\beta + \pi)$;

therefore $\Sigma \sin(2n+1)\alpha = 0$, $\Sigma \cos(2n+1)\alpha = 0$.

In the case of three points which will evidently be at the corners of an equilateral triangle, we have $\gamma - \beta = \beta - \alpha = \frac{2}{3}\pi$; the corresponding equations, which present no difficulty, are

$$\Sigma \sin(3n \pm 1)\alpha = 0, \quad \Sigma \cos(3n \pm 1)\alpha = 0.$$

10398. (Professor WOLSTENHOLME, M.A., Sc.D.)—An equilateral triangle is drawn circumscribing a given parabola $y^2 = 4ax$; prove that the envelope of its circumscribed circle is a hyperbola $3(x+a)^2 - y^2 = 12a^2$ similar to the hyperbola which is the locus of the angular points of the triangle. Of course the envelope of the inscribed circle is the similar hyperbola $3(x+a)^2 - y^2 = 3a^2$. [The locus of the angular points is well known to be the hyperbola $(3x+5a)^2 - 3y^2 = 16a^2$.]

Solution by Rev. T. GALLIERS, M.A.; G. G. STORR, M.A.; *and others.*

Let the equations of the sides of the triangle be

$$x - m_1 y + a m_1^2 = 0; \quad x - m_2 y + a m_2^2 = 0; \quad x - m_3 y + a m_3^2 = 0;$$

then, if (X_1, Y_1), (X_2, Y_2), (X_3, Y_3) be the angular points,

$$\left.\begin{aligned} X_1 &= a m_2 m_3 \\ Y_1 &= a(m_2 + m_3) \end{aligned}\right\}, \quad \left.\begin{aligned} X_2 &= a m_3 m_1 \\ Y_2 &= a(m_3 + m_1) \end{aligned}\right\}, \quad \left.\begin{aligned} X_3 &= a m_1 m_2 \\ Y_3 &= a(m_1 + m_2) \end{aligned}\right\}.$$

Hence, if $m_1 + m_2 + m_3 = S$, $m_2 m_3 + m_3 m_1 + m_1 m_2 = \Pi$, $m_1 m_2 m_3 = P$,

we may show that the orthocentre C is $\{-a, a(S+P)\}$, and the circumcentre O is $\{\frac{1}{2}a(\Pi+1), \frac{1}{2}a(S-P)\}$. The triangle being equilateral, O and C coincide; thus $\Pi = -3$, and (1) $S = -3P$; hence O is $(-a, -2aP)$; Now, R being circumradius,

$$R^2 = a^2(1 + m_2 m_3)^2 + a^2(\tfrac{1}{2}S - m_1)^2 = \&c. = \&c.;$$

therefore $3R^2 = 12a^2(P^2+1)$, or $R^2 = 4a^2(P^2+1)$. Thus the equation of the circumcircle is $x^2 + 2ax + y^2 + 4aP \cdot y = 3a^2$.....(2),

which has no envelope.

If the right-hand side of (2) had been $a^2(P^2+4)$, the envelopes would have been the curves given in the Question, viz.:

$$3(x+a)^2 - y^2 = 12a^2 \text{.....(3).}$$

If the envelope of the circumcircle were (3), the envelope of the incircle, being that of a concentric circle of half the radius, would of course be $3(x+a)^2 - y^2 = 3a^2$. To find the locus of the angular points, of (X_1, Y_1) suppose. We have

$$m_1 X_1 = aP, \quad Y_1 = -a(3P + m_1) \text{.....} (1),$$

also, since $m_2 m_3 + m_3 m_1 + m_1 m_2 = +\Pi = -3$,

$$X_1 + m_1 \cdot Y_1 = -3a \text{.....} (2).$$

From (1) and (2) we obtain $(3x+5a)^2 - 3y^2 = 16a^2$.

10578. (W. J. GREENSTREET, M.A.)—A circle passes through a fixed point on an ellipse, and has its centre on the same curve; prove that (1) the circle envelopes a quartic; (2) the quartic has a cusp at the fixed point; and (3) the tangent at the cusp is a normal to the ellipse.

Solution by Professor ANDERSON; R. KNOWLES, B.A.; *and others.*

Take the given point on the ellipse for origin, and let the axes of coordinates be parallel to the principal axes of the ellipse. Then, if (x', y') be the coordinates of the given point with reference to these latter axes, and (α, β) the centre the equations of the ellipse and circle are

$$b^2(x^2+2xx')+a^2(y^2+2yy')=0, \quad x^2+y^2-2\alpha x-2\beta y=0.$$

Hence we have $\quad b^2(\alpha^2+2\alpha x')+a^2(\beta^2+2\beta y')=0.$

The equation of the envelope is

$$(x^2+y^2+2xx'+2yy')^2-4(a^2x^2+b^2y^2)=0;$$

hence the origin is a double point on this curve, the tangents at it being

$$(xx'+yy')^2-a^2x^2-b^2y^2=0,$$

which, by help of the condition $b^2x'^2+a^2y'^2=a^2b^2$, reduces to

$$(a^2y'x-b^2x'y)^2=0.$$

The tangents being coincident, the origin is a cusp, whose tangent, $a^2y'x-b^2x'y=0$, coincides with the normal to the ellipse at the point $x'y'$.

10561. (Professor ZERR.) — If F = (force) $=\mu r^{-5}(5r^2+8c^2)$, and a particle be projected from an apse at a distance c with the velocity from infinity, show that the equation of the orbit is $r=c\cos\frac{1}{2}\theta$.

Solution by Professor ANDERSON, M.A.; D. E. SHORTO; *and others.*

Since the particle is projected with the velocity from infinity, the velocity in any position is given by

$$\tfrac{1}{2}v^2=\int_r^\infty\left(\frac{5\mu}{r^3}+\frac{8\mu c^2}{r^5}\right)dr, \text{ or } v^2=\frac{2\mu}{r^2}\left(\frac{5}{2}+\frac{2c^2}{r^2}\right);$$

and, since $vp=h$, twice the areal velocity of the particle, we have

$$h^2=9\mu; \quad \text{therefore } \frac{1}{p^2}=\frac{2}{9r^2}\left(\frac{5}{2}+\frac{2c^2}{r^2}\right),$$

which is the p and r equation of the orbit described. The stated equation in polar coordinates may be obtained in the usual way.

10484. (Professor SYÈCHNIKOFF.)—Résoudre les équations

$$x+y+z=a, \quad x^2+y^2+z^2=a^2+2b^2, \quad x^3+y^3+z^3=a^3.$$

Solution by D. T. GRIFFITHS; J. BOLAND, M.A.; *and others.*

It is obvious that x, y, z are the roots of the cubic $t^3-at^2-b^2t+ab^2=0$; *i.e.*, of $(t-b)(t+b)(t-a)=0$; hence the different solutions are

$$\begin{vmatrix} x= & a \\ y= & b \\ z= & -b \end{vmatrix} \quad \begin{vmatrix} x= & a \\ y= & -b \\ z= & b \end{vmatrix} \quad \begin{vmatrix} x= & b \\ y= & a \\ z= & -b \end{vmatrix} \quad \begin{vmatrix} x= & b \\ y= & -b \\ z= & a \end{vmatrix} \quad \begin{vmatrix} x= & -b \\ y= & a \\ z= & b \end{vmatrix} \quad \begin{vmatrix} x= & -b \\ y= & b \\ z= & a. \end{vmatrix}$$

3481. (Professor CAYLEY, F.R.S.)—Find, in the Hamiltonian form t

$$\frac{d\eta}{dt} = \frac{d\mathrm{H}}{d\varpi}, \quad \frac{d\varpi}{dt} = -\frac{d\mathrm{H}}{d\eta}, \text{ \&c.},$$

the equations for the motion of a particle acted on by a central force.

Solution by Professor SEBASTIAN SIRCOM, M.A.

We have $\mathrm{T} = \frac{1}{2}(\dot{r}^2 + r^2\dot{\theta}^2)$, $\mathrm{U} = -\int \mathrm{P}dr$;

then, writing $\eta = r$, $\varpi = \frac{d\mathrm{T}}{d\dot{r}} = \dot{r}$, $\eta_1 = \theta$, $\varpi_1 = \frac{d\mathrm{T}}{d\dot{\theta}} = r^2\dot{\theta}$,

we have $\mathrm{H} = \mathrm{T} - \mathrm{U} = \frac{1}{2}\left(\varpi^2 + \frac{\varpi_1^2}{\eta^2}\right) + \int \mathrm{P}dr$,

and the Hamiltonian equations are

$$\frac{d\eta}{dt} = \varpi, \quad \frac{d\varpi}{dt} = \frac{\varpi_1^2}{\eta^3} - \mathrm{P}; \quad \frac{d\eta_1}{dt} = \frac{\varpi_1}{\eta^2}, \quad \frac{d\varpi_1}{dt} = 0;$$

or, restoring r, θ, we have

$$dt = \frac{d\dot{r}}{r\dot{\theta}^2 - \mathrm{P}} = \frac{dr}{\dot{r}} = -\frac{d(r^2\dot{\theta})}{0} = \frac{d\theta}{\dot{\theta}},$$

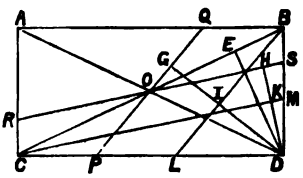
the first and third of these giving the ordinary equations.

10331. (D. BIDDLE.)—A rectangle of given dimensions is divided by a random straight line. Find the average ratio which the smaller portion bears to the larger.

Solution by the PROPOSER.

Let $AB = a$, $BD = b$; draw DE perpendicular to BC, and let

$$\angle EDC = \gamma;$$



then, for the positions (extending through a quadrant) in which D is the apex of the smaller portion cut off, the perpendicular from D to the random line forms with CD an angle which is either $< \gamma$ or $> \gamma$. Let PQ, RS be parallel to two representative random lines thus differentiated, and draw further parallels, BL, CM; also, let x = the length of the perpendicular from D to the random line. Then, if $x < \mathrm{DI}$ (or DK), the lesser segment is triangular; but if $\mathrm{DG} > x > \mathrm{DI}$ or $\mathrm{DH} > x > \mathrm{DK}$, the lesser segment is quadrilateral.

Now $\mathrm{DI} = b\sin\theta$, $\mathrm{DK} = a\cos\theta'$, $\mathrm{DG} = \frac{1}{2}(a\cos\theta + b\sin\theta)$,

and likewise $\mathrm{DH} = \frac{1}{2}(a\cos\theta' + b\sin\theta')$.

Moreover, $\triangle BLD = \frac{1}{2}b^2 \tan\theta$, and $\triangle CMD = \frac{1}{2}a^2 \operatorname{cotan}\theta'$;

whilst $BL = b\sec\theta$, and $CM = a\operatorname{cosec}\theta'$.

Where the n ent is triangular, its area is given by

$$\tfrac{1}{2}x^2(\tan\theta + \operatorname{cotan}\theta) = \tfrac{1}{2}x^2 / (\sin\theta\cos\theta);$$

and where it is quadrilateral, by

$$\tfrac{1}{2}b^2\tan\theta + (x - b\sin\theta)\, b\sec\theta = (bx - \tfrac{1}{2}b^2\sin\theta)/\cos\theta,$$

or by $\frac{1}{2}a^2\operatorname{cotan}\theta' + (x - a\cos\theta')\, a\operatorname{cosec}\theta' = (ax - \frac{1}{2}a^2\cos\theta')/\sin\theta'$.

Consequently, as A, B, C are similarly placed with D, the required mean is

$$\begin{aligned}
&\frac{2}{\pi}\left(\frac{2}{a\cos\theta + b\sin\theta}\right)\left\{\int_0^{\gamma}\int_0^{b\sin\theta}\frac{w^2}{2ab\sin\theta\cos\theta - x^2}\,d\theta\,.\,dx\right.\\
&\quad + \int_0^{\gamma}\int_{b\sin\theta}^{\frac{1}{2}(a\cos\theta + b\sin\theta)}\frac{2bx - b^2\sin\theta}{2ab\cos\theta + b^2\sin\theta - 2bx}\,d\theta\,.\,dx\\
&\quad + \int_{\gamma}^{\frac{1}{2}\pi}\int_0^{a\cos\theta}\frac{x^2}{2ab\sin\theta\cos\theta - x^2}\,d\theta\,.\,dx\\
&\quad \left. + \int_{\gamma}^{\frac{1}{2}\pi}\int_{a\cos\theta}^{\frac{1}{2}(a\cos\theta + b\sin\theta)}\frac{2ax - a^2\cos\theta}{2ab\sin\theta + a^2\cos\theta - 2ax}\,d\theta\,.\,dx\right\}\\
&= \frac{2}{\pi}\int_0^{\gamma}\left(\frac{b\sin\theta - 2}{2}\log\frac{ab\cos\theta}{2ab\cos\theta - b^2\sin\theta} + \frac{a\cos\theta}{a\cos\theta - b\sin\theta}\right.\\
&\qquad \left. - \frac{2(2ab\sin\theta\cos\theta)^{\frac{1}{2}}}{a\cos\theta + b\sin\theta}\log\frac{(2ab\sin\theta\cos\theta)^{\frac{1}{2}} - b\sin\theta}{(2ab\sin\theta\cos\theta - b^2\sin^2\theta)^{\frac{1}{2}}}\right)d\theta\\
&\quad + \frac{2}{\pi}\int_{\gamma}^{\frac{1}{2}\pi}\left(\frac{a\cos\theta - 2}{2}\log\frac{ab\sin\theta}{2ab\sin\theta - a^2\cos\theta} - \frac{b\sin\theta}{a\cos\theta + b\sin\theta}\right.\\
&\qquad \left. - \frac{2(2ab\sin\theta\cos\theta)^{\frac{1}{2}}}{a\cos\theta + b\sin\theta}\log\frac{(2ab\sin\theta\cos\theta)^{\frac{1}{2}} - a\cos\theta}{(2ab\sin\theta\cos\theta - a^2\cos^2\theta)}\right)d\theta,
\end{aligned}$$

which does not seem to work out easily.

2665. (Professor CROFTON, F.R.S.) — 1. Given any closed convex boundary, without salient points: if dS be any element of the external area; T, T′ the tangents drawn from dS to the boundary; θ the angle they contain; and if the integration extend over the whole external area; then prove that

$$\iint \frac{\sin\theta}{TT'}\,dS = 2\pi^2.$$

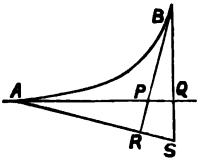
2. If the integration extend over the annular space between the proposed boundary and an outer line along which θ has a constant value (α), the value of the ntegral is $2\pi(\pi - \alpha)$.

3. If the given boundary has salient points, then for every such point where the bounding line changes direction abruptly through an angle A, we must subtract $\frac{1}{2}A^2$ from the above values: then the value is $2\pi^2 - \frac{1}{2}\Sigma A$. For instance, for a regular polygon of n sides, the value will be

$$2\pi^2(1 - 1/n).$$

Solution by Professors SIRCOM *and* ANDERSON.

1. Let BR, BS be two consecutive tangents at B making an angle $d\theta$; AQ, AS consecutive tangents at A making an angle $d\theta_1$;



$$\text{BPQ} = \theta,$$

$$d\text{S} = \text{area PQRS} = \text{PQ} \cdot \text{PR} \sin\theta;$$

then, since $\text{PQ} \sin\theta = \text{T}d\theta$, $\text{PR} \sin\theta = \text{T}'d\theta_1$,

the integral becomes $\iint d\theta\, d\theta_1$.

We have to integrate first over the area RBS; this gives 0, θ for the limits of θ_1; then the limits 0, 2π for θ give the integral over the whole external area. Hence the value of the integral is $2\pi^2$.

2. If the integration is extended over the annular space in question the limits for θ_1 are 0, $\pi - \alpha$, since we are taking the external angle between the tangents; and the value is $2\pi(\pi - \alpha)$.

3. The contour which has one salient point is completely described if, in taking the sum of $\theta\, d\theta$, the tangent starts at the angle A and turns to 2π, and the integral is $\int_A^{2\pi} \theta\, d\theta = 2\pi^2 - \frac{1}{2}\text{A}^2$; so for every other such salient point.

The integral $\iint \frac{\rho\rho'}{\text{TT}'} \sin\theta\, d\text{S}$ is easily found in the same manner to be $\frac{1}{2}\text{L}^2$, where L is the perimeter.

[The PROPOSER's solution is given in the *Philosophical Transactions* for 1868; but the above is a direct proof.]

10607. (W. J. C. SHARP, M.A.)—If Δ_1, Δ_2, Δ_3, Δ_4 be the areas of the faces of the tetrahedron of reference, and (x, y, z, w) the quadruplanar coordinates of any point P on $\Delta_1/x + \Delta_2/y + \Delta_3/z + \Delta_4/w = 0$, show that the feet of the perpendiculars from P upon the faces li in the same plane. Find the equation of this plane.

Solution by Professor SEBASTIAN SIRCOM, M.A.

Let (x_1, y_1, z_1, w_1) be the coordinates of P; X, Y, Z, W the feet of the perpendiculars from P on the faces of the tetrahedron of reference. The equation of the surface being $\Delta_1 yzw + \ldots = 0$, six times the volume of the tetrahedron PYZW is equal to $yzw \sin(yzw)$, where (yzw) is the solid angle subtended at P by YZW. But the plane angles YPZ, &c. are the supplements of the angles between the corresponding planes of the tetrahedron of reference if P is within the tetrahedron; hence $\sin(yzw)$ is equal to

$$\begin{vmatrix} 1, & -\cos(yz), & -\cos(yw) \\ -\cos(yz), & 1, & -\cos(zw) \\ -\cos yw, & -\cos zw, & 1 \end{vmatrix};$$

therefore $\propto \Delta_1$; and we shall also obtain $\sin(yzw) \propto \Delta_1$ if P is without. Hence, the sum of the tetrahedra being 0 by the equation of the surface, X, Y, Z, W lie in a plane. The coordinates of Y are

$$x_1 + y_1\cos(xy), \quad 0, \quad z_1 + y_1\cos(yz), \quad w_1 + y_1\cos(yw),$$

so for Z and W, and the equation of the plane is

$$\begin{vmatrix} x, & y, & z, & w \\ x_1 + y_1\cos(yx), & 0, & z_1 + y_1\cos yz, & w_1 + y_1\cos(yw) \\ x_1 + z_1\cos(zx), & y_1 + z_1\cos(zy), & 0, & w_1 + z_1\cos(zw) \\ x_1 + w_1\cos(wx), & y + w_1\cos(wy), & z_1 + w_1\cos wz, & 0 \end{vmatrix} = 0.$$

To reduce the coefficient of y, we observe that

$$-\Delta_1 + \Delta_2\cos(xy) + \Delta_3\cos(xz) + \Delta_4\cos(xw) = 0,$$

and three others; whence, eliminating successively Δ_1, Δ_3, Δ_4 from the equation of the surface, and the first, third, and fourth of these, we have

$$\Delta_2 z_1 w_1 \{x_1 + y_1\cos(xy)\} + \Delta_3 y_1 w_1 \{x_1 + z_1\cos(xz)\} + \Delta_4 y_1 z_1 \{x_1 + w_1\cos(xw)\} = 0,$$

and two others; hence, multiplying the rows of the coefficient of y by $\Delta_2 z_1 w_1$, $\Delta_3 y_1 w_1$, $\Delta_4 y_1 z_1$, and adding, we have

$$\frac{\Delta_1 y_1}{\Delta_2 x_1} \times \begin{vmatrix} 0, & z_1 + x_1\cos(xz), & w_1 + x_1\cos(xw) \\ x_1 + z_1\cos(xz), & 0, & w_1 + z_1\cos(zw) \\ x_1 + w_1\cos(wx), & z_1 + w_1\cos(wz), & 0 \end{vmatrix}.$$

The coefficients of z, w being reduced in like manner, the equation of the plane in question is

$$\frac{xx_1}{\Delta_1}\{[w_1 + z_1\cos(zw)][y_1 + w_1\cos(wy)][z_1 + y_1\cos(yz)] + [z_1 + w_1\cos(zw)][w_1 + y_1\cos(wy)][y_1 + z_1\cos(yz)]\} + \&c. = 0.$$

[Mr. SHARP remarks that this Question is a special case of the theorem enunciated in Art. XXVI. of the second paper on "Simplicissima in Space of n-dimensions" (*Proc. Lond. Math. Soc.*, Vol. XIX., p. 449). In a further paper, read before the Society at its last meeting, he has returned to the subject, and has shown that, using Prof. SIRCOM's notation, the content of the simplicissimum PYZW ... is $\frac{n^{n-2}}{\{(n-1)!\}^2}\frac{y_1 z_1}{\Delta_2\Delta_3}\nabla^{n-1}$ in space of n dimensions, and therefore in ordinary space the tetrahedron $\mathrm{PYZW} = \frac{3}{4}\frac{y_1 z_1 w_1}{\Delta_2\Delta_3\Delta_4}\nabla^2$. This he had proved before he obtained the general result as follows:—

Denoting the sides and angles of the spherical triangle determined by the solid angle opposite Δ_1 by a, b, c and A, B, C, if $a = \mathrm{ZPW}$, $\beta = \mathrm{WPY}$, $\gamma = \mathrm{YPZ}$, $\alpha = \pi - \mathrm{A}$, $\beta = \pi - \mathrm{B}$, $\gamma = \pi - \mathrm{C}$, and

$$(\mathrm{PYZW})^2 = \tfrac{1}{36}\mathrm{PY}^2.\mathrm{PZ}^2.\mathrm{PW}^2 \begin{vmatrix} 1, & \cos\gamma, & \cos\beta \\ \cos\gamma, & 1, & \cos\alpha \\ \cos\beta, & \cos\alpha, & 1 \end{vmatrix}$$ (*Proc. Lond. Math. Soc.*, Vol. XVIII., p. 333)

$$= \tfrac{1}{36} y_1^2 z_1^2 w_1^2 \frac{1}{\sin^2 a + \sin^2 b + \sin^2 c} \begin{vmatrix} 1, & \cos c, & \cos b \\ \cos c, & 1, & \cos a \\ \cos b, & \cos a, & 1 \end{vmatrix}^2,$$

and $$\mathrm{PYZW} = \frac{y_1 z_1 w_1}{6 \sin a \sin b \sin c} \begin{vmatrix} 1, & \cos c, & \cos b \\ \cos c, & 1, & \cos a \\ \cos b, & \cos a, & 1 \end{vmatrix}$$

$$= \frac{y_1 z_1 w_1}{6} \cdot \frac{36 \mathrm{V}^2}{\mathrm{AB}^2 . \mathrm{AC}^2 . \mathrm{AD}^2} \cdot \frac{1}{\sin a \sin b \sin c} = \tfrac{1}{4} \frac{y_1 z_1 w_1}{\Delta_2 \Delta_3 \Delta_4} \mathrm{V}^2,$$

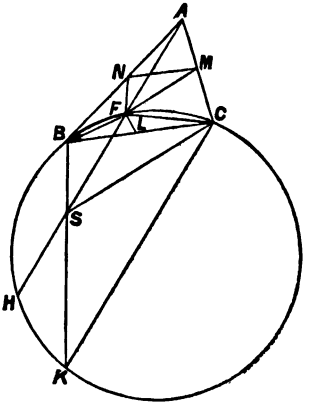
from which the rest follows at once.

This result furnishes a geometrical interpretation of all equations of the form $\mathrm{A}/x + \mathrm{B}/y + \mathrm{C}/z + \mathrm{D}/w = 0$ as the corresponding result in plane geometry (see Vol. XLIX., App. I., p. 131, and Questions 9817 and 9818) does of those of the form $\mathrm{A}/x + \mathrm{B}/y + \mathrm{C}/z = 0$.]

10495. (Professor MORLEY.)—Let ABC be a triangle, and let F be a focus of the maximum inscribed ellipse. Denoting these points by the complex variables z_1, z_2, z_3, z, prove that (1) $\Sigma 1/(z - z_r) = 0$, $(r = 1, 2, 3)$; and hence (2) if AF meets the circle FBC again at H, $\mathrm{FH} = 2\mathrm{AF}$.

Solution by Professors ANDERSON, SARKAR, *and others.*

1. Let L, M, N be the middle points of the sides and points of contact of the ellipse, and let F be a focus. Draw CK parallel to AF, meeting the circle in K, and join B, K intersecting AH in the point S.



Denote $z_1 - z$ by α, $z_2 - z$ by β, and $z_3 - z$ by γ. Then the point M referred to F as origin will be $\frac{1}{2}(\alpha + \gamma)$, and the point N, $\frac{1}{2}(\alpha + \beta)$.

Since $\angle \mathrm{MFA} = \mathrm{AFN}$, we have

$$\frac{\mathrm{FN}}{\mathrm{FA}} \alpha^2 = \frac{1}{4} \frac{\mathrm{FA}}{\mathrm{FM}} (\alpha + \beta)(\alpha + \gamma),$$

or

$$\frac{4\mathrm{FN} . \mathrm{FM} - (\mathrm{FA})^2}{(\mathrm{FA})^2} \alpha^2 = \alpha\beta + \beta\gamma + \gamma\alpha.$$

In like manner, $$\frac{4\mathrm{FL} . \mathrm{FN} - (\mathrm{FB})^2}{(\mathrm{FB})^2} \beta^2 = \alpha\beta + \beta\gamma + \gamma\alpha,$$

and $$\therefore \frac{4\mathrm{FN} . \mathrm{FM} - (\mathrm{FA})^2}{(\mathrm{FA})^2} \alpha^2 = \frac{4\mathrm{FL} . \mathrm{FN} - (\mathrm{FB})^2}{(\mathrm{FB})^2} \beta^2 = \frac{4\mathrm{FL} . \mathrm{FM} - (\mathrm{FC})^2}{(\mathrm{FC})^2} \gamma^2.$$

The coefficients of α^2, β^2, γ^2, therefore, separately vanish, and

$$\alpha\beta + \beta\gamma + \gamma\alpha = 0, \quad \text{or} \quad \frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} = 0,$$

which proves the first part of the question.

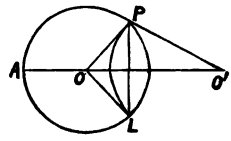
2. We have now $4FN \cdot FM = (AF)^2$. Also, since CK is parallel to AF and $\angle CKB = \angle NFA$, BK is parallel to NF, and therefore CS to MF; therefore $\angle BSC = 2\angle SKC$; therefore SC = SK and SH = SF; also BS = 2NF and CS = 2FM, therefore

$$BS \cdot SK = BS \cdot SC = 4FM \cdot FN = (FA)^2,$$

therefore $(SH)^2 = (FA)^2$, and therefore FH = 2AF.

9447. (S. Tebay, B.A.)—Draw a circle to trisect the circumference of a given circle and cut it at right angles.

Solution by W. J. Greenstreet, M.A.; J. O'Regan; *and others.*



Let O be any circle, PT any tangent to it; draw OO′ at 60° to OP; then the circle with centre O′, radius O′P, will cut O at right angles at P. For let it cut the circle O again at L; then we have

$\angle POO' = O'OL = 60°$ and $AOP = 120°$;

therefore L is a point of trisection.

[The analagous problem for trisection of *area* forms Question 10705.]

10637. (Major-General P. O'Connell.)—Solve the equation

$$x^6 + 378x^5 + 38189x^4 + 492368x^3 - 572554x^2 + 213720x - 26352 = 0.$$

Solution by the Proposer.

Let $x = \frac{1}{3}y$; then, multiplying by 3^6, and putting $y = y' + 1{\cdot}1$, thereafter $y' = 10/z$; then, multiplying by 10^6, and dividing by z^3, we have

$$(z^3 + z^{-3}) + 11406\,(z^2 + z^{-2}) + 34995615\,(z + z^{-1}) + 14819968420 = 0.$$

If $u = z + z^{-1}$, we have, after making transfers to complete u^3 and u^2,

$$u^3 + 11406u^2 + 34995612u + 14819945608$$
$$\equiv (u + 5902)(u + 5002)(u + 502) = 0.$$

Substituting for u its value $10y' + 1/(10y)$, and dividing by 100, we have for solution the three quadratics

$$y'^2 + 590{\cdot}2y' + {\cdot}01 = 0, \quad y'^2 + 500{\cdot}2y' + {\cdot}01 = 0, \quad y'^2 + 50{\cdot}2y' + {\cdot}01 = 0;$$

and, since $x = \frac{1}{3}y = \frac{1}{3}(y' + 1{\cdot}1)$, the required six values of x can be obtained as surds.

[This equation is taken from the *Mathematician*, Vol. I., p. 165. It is the last of three equations said to have been given by Mr. Lockhart for solution by British students in 1841. Several of its roots coincide to some

places of decimals. Mr. LOCKHART's method of forming numerical equations having nearly equal roots is stated in the article from which this equation is taken. It will be found, however, that this equation can be transformed into another, the solution of which only demands that of one cubic and three quadratics. The roots can be given in the form of surds, and the solution suggests a somewhat different method of forming equations having nearly equal roots from Mr. Lockhart's. It leads, in fact, to expressions for the coefficients of such equations. In the 2nd edition of YOUNG's *Algebraic Equations*, this equation is solved by the aid of STURM's *Theorem* and HORNER's *Method*, and the solution occupies fully nineteen large pages.]

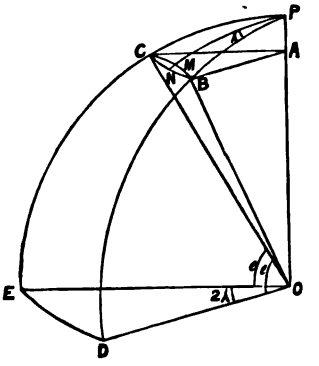
9999. (R. H. W. WHAPHAM, B.A.)—Two ports B and C are in the same parallel of latitude (C being west of B); their common latitude being l, and their difference of longitude 2λ. Two ships set out to sail from B to C, one due West and the other along the great circle through B and C; if they sail uniformly, and both arrive at C at the same time, prove that their distance apart when they have gone half-way is

$$r\left\{\tfrac{1}{2}\pi - l - \tan^{-1}(\cot l \cos\lambda)\right\},$$

where r is the radius of the earth, and l expressed in circular measure.

Solution by the PROPOSER; Professor AIYAR, *and others.*

Let BMC and BNC be the great and small circles joining B and C, and draw PMN perpendicular to BMC; it will be perpendicular to BNC, and will bisect BMC and BNC.



Now arc PN = PB = $\frac{1}{2}\pi - l$;

$\therefore$ length of PN = $r(\frac{1}{2}\pi - l)$.

Again, in the right-angled triangle PMB, we get

$\tan PM = \tan PB \cos\lambda = \cot l \cos\lambda$;

$\therefore$ $PM = r\tan^{-1}(\cot l \cot\lambda)$;

therefore the required distance

$= MN = PN - PM$

$= r\left\{\tfrac{1}{2}\pi - l - \tan^{-1}(\cot l \cos\lambda)\right\}$.

10093. (R. H. W. WHAPHAM, B.A.)—From any point P on the evolute of a parabola are drawn normals PQ, PQ′ to the parabola, Q being on the same side of the axis as P; prove that the distance from the vertex of the evolute to the foot of the perpendicular from P on the axis

is three times the part of the axis intercepted between this latter point and the normal PQ, and that it is also three times the part of the axis intercepted between the vertex of the evolute and the normal PQ.

Solution by the PROPOSER; Professor CHAKRAVARTI; *and others.*

Let equation to parabola be

$$y^2 = 4ax,$$

then evolute is

$$27ay^2 = 4(x-2a)^3,$$

and if the coordinates of P be $(2a+3\delta, \epsilon)$, the equation to parabola is normal to

$$am^3 - (x-2a)m + y = 0;$$

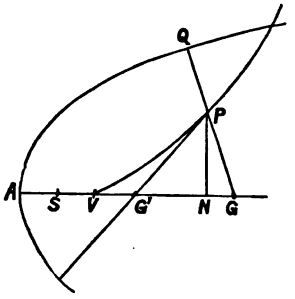
and if this pass through P, we have

$$am^3 - 3\delta m + \epsilon = 0.$$

Also, since P is on the evolute,

$$27a\epsilon^2 = 4(3\delta)^3 \text{ or } a\epsilon^2 = 4\delta^3,$$

which is condition that the preceding equation should have two roots equal.



Let m_1 be tangent of inclination of PQ to axis; then, if m_2 is the double root of (iv.), m_2 will be the tangent of the inclination of PQ′ to the axis. Therefore, from (iv.), $m_1 + 2m_2 = 0$, $2m_1m_2 + m_2^2 = -3\delta/a$, $m_1m_2^2 = -\epsilon/a$; therefore $m_2 = \sqrt{3}/\sqrt{a} = \epsilon/2\delta$, from (iii.); $m_1 = -\epsilon/\delta$. But

$$PN/NG = -m_1 = \epsilon/\delta \text{ and } PN = \epsilon, \text{ therefore } NG = \delta;$$

and $$PN/NG' = m_2 = \epsilon/2\delta, \text{ therefore } NG' = 2\delta;$$

but $VN = 3\delta$, therefore $VG' = \delta$, therefore $VN = 3NG = 3VG'$.

9088. (A. R. JOHNSON, M.A.)—Express (1), in terms of the fundamental covariants and invariants of a system of three conics, the covariant which, equated to zero, gives the locus of the point the tangents from which to the three conics form a system in involution; also, find (2) the equation to the scroll generated by the lines touching each of three quadrics; and (3) express, in terms of the fundamental invariants of three quadrics, the invariant whose vanishing is the condition that the three quadrics should have a common self-conjugate pentahedron.

Solution by the PROPOSER.

I. *General considerations.*—When we know the fundamental invariants of a system of quantics of the same order, then the *combinant* invariants of the system are easily found. For we have only to write down the symmetrical groups of sums of products of the invariants containing the coefficients in the required degree, and adding all the groups, each multiplied by an arbitrary constant, to differentiate the sum with respect

to one set of coefficients while multiplying by the corresponding components of another set of coefficients, and to equate the result to zero. We shall then have, in general, more than enough equations to determine the arbitraries. If these are all consistent, they give the required combinant; if they are inconsistent the deduction is that the combinant cannot be expressed in terms of the fundamental invariants, though its square or some higher power can be.

II. It is known that a system of three $2m$-ary quadrics cannot have a self-conjugate $(3m-1)$-hedron, unless an invariant of the mth order in the coefficients of each vanishes. This invariant is manifestly a combinant. When $m = 1$, the invariant vanishing is the condition that the three pairs of lines should form a system in involution. Denote the pairs by (1), (2), (3); and let $\cdot 1^2 \cdot$ denote the discriminant of the first pair, and let differentiation with respect to the coefficients of (1) and simultaneous multiplication with respect to the coefficients of (2) be denoted by $({}_2\partial_1)$.

Then the invariants of the system are

$$\cdot 1^2 \cdot, \quad \cdot 12 \cdot = ({}_2\partial_1) \cdot 1^2 \cdot, \quad \cdot 2^2 \cdot = \tfrac{1}{2} ({}_3\partial_1)^2 \cdot 1^2 \cdot.$$

[In general, in the case of three m-ary quadrics they are

$$\cdot 1^m \cdot, \quad \cdot 1^{m-1} 2 \cdot = ({}_2\partial_1) \cdot 1^m \cdot, \quad \cdot 1^{m-2} 2^2 \cdot = \frac{1}{2!} ({}_2\partial_1)^2 \cdot 1^m \cdot, \text{ \&c.};$$

$$1^{m-r} \cdot 2^r = \frac{1}{r!} ({}_2\partial_1)^r \cdot 1^m \cdot, \quad \cdot 1^{m-p-q} 2^p 3^q \cdot = \frac{1}{p!\, q!} ({}_2\partial_1)^p ({}_3\partial_1)^q \cdot 1^m \cdot,$$

where $\cdot 1^m \cdot$ is the discriminant of (1).

Then $\quad ({}_1\partial_3) \cdot 1^p 2^q 3^r \cdot \ (p+q+r = m \text{ and } r \neq 0)$

is $(p+1) \cdot 1^{p+1} 2^q 3^{r-1}$, and is zero when $r = 0$.]

It is impossible to express any combinant of the first degree in the coefficients in terms of the fundamental invariants.

Let us therefore find the *combinant* involving the coefficients in the second degree. This is

$$A \cdot 1^2 2^2 \cdot 3^2 \cdot + B \left\{ \cdot 1^2 \cdot (\cdot 23 \cdot)^2 + \cdot 2^2 \cdot (\cdot 31 \cdot)^2 + \cdot 3^2 \cdot (\cdot 12 \cdot)^2 \right\} + C \cdot 23 \cdot 31 \cdot 12 \ldots (a).$$

Operating with $({}_1\partial_3)$, we have identically

$$0 = A \cdot 1^2 \cdot 2^2 \cdot 13 \cdot + B \left\{ 2 \cdot 1^2 \cdot 12 \cdot 23 \cdot + 4 \cdot 2^2 \cdot 1^2 \cdot 13 \cdot + \cdot 13 \cdot (\cdot 12 \cdot)^2 \right\} + C \left\{ (\cdot 12 \cdot 1^2 \cdot 3 \cdot + 2 \cdot 1^2 \cdot 23 \cdot 12 \cdot \right\},$$

so that $\quad A = 4, \quad B = -1, \quad C = +1,$

and the result of substitution in (a) of these values is the only combinant containing the coefficients in the second degree. It is therefore the square of the combinant we are seeking. Calling this last C, we have, then, $C^2 = 4 \cdot 1^2 \cdot 2^2 \cdot 3^2 \cdot - \left\{ \cdot 1^2 \cdot (\cdot 23 \cdot)^2 + \cdot 2^2 \cdot (\cdot 31 \cdot)^2 + \cdot 3^2 \cdot (\cdot 12 \cdot)^2 \right\} + \cdot 23 \cdot 31 \cdot 12.$

III. $m = 2$.—A similar process gives, in the case of three solid quadrics,

$$\begin{aligned} C^2 = {} & 144 \cdot 1^4 \cdot 2^4 \cdot 3^4 \cdot - 36 \cdot 1^4 \cdot 2^3 3 \cdot 2^3 3 \cdot + 12 \cdot 1^4 \cdot (\cdot 2^3 3^2 \cdot)^2 \\ & + 9 \cdot 1^3 2 \cdot 12^2 3 \cdot 2^3 3 \cdot - 6 \cdot 1^3 2 \cdot 12^3 2 \cdot 2^3 3^2 \cdot + 9 \cdot 1^3 2 \cdot 13^2 \cdot 2^3 3 \cdot \\ & + 4 \cdot 1^2 2^2 \cdot 1^2 2^2 \cdot 2^2 3^2 + 2 (\cdot 1^2 23 \cdot)^2 \cdot 2^2 3^2 \cdot - \cdot 1^2 23 \cdot 12 3^2 \cdot 12 3, \end{aligned}$$

where each term stands for the sum of all terms obtained from it by interchanging the symbols 1, 2, 3 in all possible ways.

The value of the square of C may be investigated in the same way in the case of the 6-ary, 8-ary, &c. quadric.

IV. *Application to the Question.*—(3) is already solved in § III.

(1) Call the conics (1), (2), (3); and let the equations to the conics be $s_{11} = 0$, $s_{22} = 0$, $s_{33} = 0$; as formed from the tangential equations. Let also $s_{23} = 0$, $s_{31} = 0$, $s_{12} = 0$ be the equations to the director conics formed in the same manner. Then with regard to the points in which the tangents to (1), (2), (3), from any point, cut any line $l = 0$, we have, in terms of the coordinates of the point,

$$\cdot 1^2 \cdot = 7^2 \cdot s_{11}, \quad \cdot 12 \cdot = 7^2 \cdot s_{12}, \text{ \&c.},$$

and we have, therefore, on substitution in σ^2 and division by l^4,

$$\mathrm{I}^2 = 4 s_{11} s_{22} s_{33} - \left\{ s_{11} s_{23}^2 + s_{22} s_{31}^2 + s_{33} s_{12}^2 \right\} + s_{23} s_{31} s_{12},$$

$\mathrm{I} = 0$ being the equation to the locus.

V. To answer (2), we first express in terms of the fundamental invariants the condition that three conics should meet at a point. This comes out as

$$\begin{aligned} \mathrm{R} = {} & (\cdot 123 \cdot)^4 - 8 (\cdot 123 \cdot)^2 \cdot 12^2 \cdot 13^2 \cdot - 72 \cdot 123 \cdot 1^3 \cdot 2^3 \cdot 3^3 \cdot \\ & + 24 \cdot 123 \cdot 1^3 \cdot 23^2 \cdot 32^2 \cdot + 16 \cdot 123 \cdot 1^2 2 \cdot 2^2 3 \cdot 13^2 \\ & + 48 \cdot 2^3 \cdot 3^3 \cdot 1^2 2 \cdot 1^3 3 \cdot - 48 \cdot 1^3 \cdot 12^2 \cdot (\cdot 23^2 \cdot)^2 + 16 (\cdot 12^2 \cdot)^2 (\cdot 13^2 \cdot)^2 \\ & - 16 \cdot 1^2 2 \cdot 1^2 3 \cdot 2^2 3 \cdot 23^2, \end{aligned}$$

where each terms stands for the sum of all terms derived from it by interchanging the symbols 1, 2, 3 in all possible ways.

Now, if a point be on the scroll, the three tangent cones through it to the three quadrics have a common generator; *i.e.*, the section of the cones by any plane have a point in common.

Now, with regard to the intersections by any plane $p = 0$ of the tangent cones from any point to the quadrics (1), (2), (3), we have, in terms of the point ($u_1 = 0$, $u_2 = 0$, $u_3 = 0$ being the equations to the quadrics),

$$\cdot 1^3 \cdot = p^3 \cdot 1^4 \cdot u_1^2 \cdot 1^2 2 \cdot = p^2 u_1 (\cdot 1^2 2 \cdot u_2 - \mathrm{S}_{12}),$$

where S_{12} is half the result of operating the tangential equation of (1) into the square of the polar of the point with respect to (2);

$$\cdot 123 \cdot = \tfrac{1}{8} p^2 (u_1 \mathrm{P}_1 + u_2 \mathrm{P}_2 + u_3 \mathrm{P}_3),$$

where P_1 is the result of operating the tangential equation of (1) on the product of the polar planes of the point with respect to (2) and (3).

The equation to the scroll is therefore found by substituting in $\mathrm{R} = 0$,

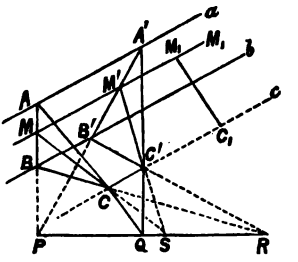
$$\cdot 1^4 \cdot u_1^2 \text{ for } \cdot 1^3 \cdot, \quad u_1 (\cdot 1^2 2 \cdot u_2 - \mathrm{S}_{12}) \text{ for } \cdot 1^2 2 \cdot,$$

and

$$\tfrac{1}{8} (u_1 \mathrm{P}_1 + u_2 \mathrm{P}_2 + u_3 \mathrm{P}_3) \text{ for } \cdot 123.$$

10570. (Professor Moritz.)—Construire un triangle sachant que les côtés passent par trois points donnés en ligne droite;—que deux des sommets se trouvent sur deux parallèles données, et qu'enfin la médiane issue du troisième sommet a une longueur connue.

Solution by J. C. St. Clair; Professor Aiyar; *and others.*



Let P, Q, R be the given collinear points, and let a, b be the given parallel lines.

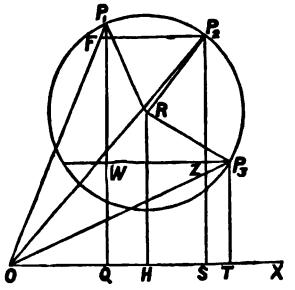
If we take any two triangles ABC, A'B'C' of which the corresponding sides meet in P, Q, R, and the vertices A, A'; B, B' lie on a and b, it is evident from a well-known theorem that c, c' lie on a fixed line c concurrent with, *i.e.* parallel to, AA', BB'. Conversely, since the mid-points M, M' lie on a fixed line m parallel to AA', CC', the lines CM, C'M' meet in a fixed point S on PQ. Hence, having by means of any triangle A'B'C' determined c and S, we have only to take any segment M_1C_1 equal to the given median line, through S draw CM parallel to C_1M_1 and draw PM meeting a, b, in A, B. Then the sides AB, BC, AC will pass through P, Q, R; and the median CM is equal to the given line.

4114. (A. Renshaw.)—If OX be a line drawn outside a circle whose centre is R, and RH the perpendicular from R on OX; also, if P_1Q, P_2S, P_3T be drawn at right angles to OX from any three points P_1, P_2, P_3 on the circumference; prove that

$$2RH\,(OS\,.\,P_1W - OQ\,.\,P_2Z - OT\,.\,P_1F) = OP_2^2\,.\,QT - OP_1^2\,.\,ST - OP_3^2\,.\,QS,$$

where P_1W, P_2Z, P_1F are the differences of the perpendiculars from P_1P_3, P_2P_3, P_1P_2, and QT, ST, QS the segments of OX between the feet of perpendiculars on OX.

Solution by D. Biddle *and* Morgan Brierley.



Let r = radius of circle, A = RH, B = OH, a_1, a_2, a_3 = vertical, and b_1, b_2, b_3 = horizontal distances from R of P_1, P_2, P_3 respectively; then

$$a_1^2 + b_1^2 = a_2^2 + b_2^2 = a_3^2 + b_3^2 = r^2;$$

$$P_1W = a_1 - a_3,$$

$$P_2Z = a_2 - a_3, \quad P_1F = a_1 - a_2,$$

$$QS = b_2 - b_1, \quad QT = b_3 - b_1,$$

$$ST = b_3 - b_2, \quad OQ = B + b_1,$$

$$OS = B + b_2, \quad OT = B + b_3.$$

$$OP_1^2 = (A + a_1)^2 + (B + b_1)^2 = M + 2Aa_1 + 2Bb_1,$$

$$OP_2^2 = (A + a_2)^2 + (B + b_2)^2 = M + 2Aa_2 + 2Bb_2,$$

$$OP_3^2 = (A + a_3)^2 + (B + b_3)^2 = M + 2Aa_3 + 2Bb_3;$$

where $M = A^2 + B^2 + r^2$.

Now it is easy to see that

$$2A\{(B+b_2)(a_1-a_3)-(B+b_1)(a_2-a_3)-(B+b_3)(a_1-a_2)\}$$

reduces to $2A\{b_2(a_1-a_3)-b_1(a_2-a_3)-b_3(a_1-a_2)\}$ .....(1),

also that $(M+2Aa_2+2Bb_2)(b_3-b_1)-(M+2Aa_1+2Bb_1)(b_3-b_2)$
$-(M+2Aa_3+2Bb_3)(b_2-b_1)$

reduces to $2A\{a_2(b_3-b_1)-a_1(b_3-b_2)-a_3(b_2-b_1)\}$ .....(2).

But (1) = (2). Hence, &c.

7251. (Professor WOLSTENHOLME, M.A., Sc.D.)—At each point P of an epicycloid (or hypocycloid) is drawn the equiangular spiral of closest contact (four-pointic), S being its pole; prove that (1) S, P are inverse points with respect to the fixed circle of the epicycloid; (2) the arc traced out by S during one complete revolution of the moving circle is

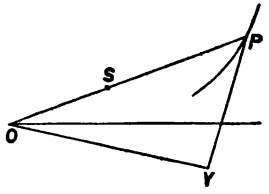
$$2a\left(1-\frac{a^2}{(a+2b)^2}\right)^{\frac{1}{2}}\log\left(\frac{(a+b)^{\frac{1}{2}}+b^{\frac{1}{2}}}{(a+b)^{\frac{1}{2}}-b^{\frac{1}{2}}}\right) \text{ for the epicycloid,}$$

and $4a\left(\frac{a^2}{(a-2b)^2}-1\right)^{\frac{1}{2}}\sin^{-1}\left(\frac{b}{a}\right)^{\frac{1}{2}}$ or $4a\left(\frac{a^2}{(2b-a)^2}-1\right)^{\frac{1}{2}}\cos^{-1}\left(\frac{b}{a}\right)^{\frac{1}{2}}$

for the hypocycloid, according as $2b <$ or $> a$; a, b being the radii of the fixed and moving circles respectively; (3) the radii of curvature at S, P of the curves are as OS : SP − OS; and (4) there will be inflexions on the locus of S whenever $OS\sqrt{2} = a$; whence by inversion it appears that, in any epicycloid, when the central distance is $a\sqrt{2}$, the circle of curvature will pass through the centre of the fixed circle.

Solution by the PROPOSER.

1. In any curve, if the equiangular spiral of closest contact be drawn at a point P, and S be its pole, the tangents at P, S to their loci are equally inclined to PS. [This was proved in the *Educational Times* many years ago.] Hence, if the locus of P be such that S always lies on OP, when O is a fixed point O, the locus of S will be the inverse with respect to O of the locus of P; for, if P, P′ be contiguous points, S, S′ the corresponding poles of the osculating spirals, a circle will go round PP′S′S, and $OP \cdot OS = OP' \cdot OS'$. Now, let α be the angle of the spiral, ρ the radius of curvature at P, ϕ the angle which the tangent at P makes with some fixed line, p the perpendicular OY from O on the tangent at P; then we have



$$PY = \frac{dp}{d\phi}, \text{ and } \frac{1}{\rho}\frac{d\rho}{d\phi} = \cot\alpha = \cot OPY = \frac{1}{p}\frac{dp}{d\phi},$$

or $\rho = kp$. Thus the theorem enunciated is true for any curve in which

the radius of curvature is a constant multiple of p. This gives the hypocycloids ($k = -\infty$ to $k = 0$); the epicycloids ($k = 0$ to $k = 1$) ($k = 1$ is the involute of a circle); and the curves whose intrinsic equation is $s = Ae^{m\phi} + Be^{-m\phi}$ [$k = 1$ to $k = +\infty$; in fact, $m = (k^2-1)^{\frac{1}{2}}$]. This includes two different families of curves, in one of which A and B have like signs, in the other unlike; any curve of one family being the evolute of some curve of the other family. To return to the epicycloid,

$$\rho \sin\alpha = \text{PS}, \quad \rho = \frac{4b\,(a+b)}{(a+2b)^2}\,p,$$

if a, b be the radii of the fixed and moving circles, $a \sin\alpha = p/r$; whence

$$\text{PS} = \frac{4b\,(a+b)}{(a+2b)^2}\,\frac{p^2}{r}; \text{ but } p^2 = \frac{(a+2b)^2}{4b\,(a+b)}\,(r^2-a^2),$$

so that $\text{PS} = r - a^2/r$, or $\text{OS} = a^2/r$, or the locus of S, is the inverse of the epicycloid with respect to the fixed circle.

2. If s', s be the arcs traced out by P, S starting from a cusp,

$$\frac{ds'}{ds} = \frac{\text{OS}}{\text{OP}} = \frac{a^2}{r^2}; \text{ but } \frac{ds}{d\theta} = (a+b)\sin\frac{a\theta}{2b},\ r^2 = (a+2b)^2 - 4b\,(a+b)\cos^2\frac{a\theta}{2b},$$

with the usual notation, or if

$$-\cos\frac{a\theta}{2b} = z, \quad \frac{ds'}{dz} = \frac{4ab\,(a+b)}{(a+2b)^2 - 4b\,(a+b)\,z^2};$$

and the whole arc corresponding to a single branch of the epicycloid

$$= 4ab\,(a+b)\int_{-1}^{+1} \frac{dz}{(a+2b)^2 - 4b\,(a+b)\,z^2} = \frac{4a\,[b\,(a+b)]^{\frac{1}{2}}}{a+2b}\log\left\{\frac{(a+b)^{\frac{1}{2}} + b^{\frac{1}{2}}}{(a+b)^{\frac{1}{2}} - b^{\frac{1}{2}}}\right\}.$$

For the hypocycloid, writing $-b$ for b, and correcting the sign of ds'/dz,

$$\frac{ds'}{dz} = \frac{4ab\,(a-b)}{(a-2b)^2 + 4b\,(a-b)\,z^2};$$

$$(1)\ a > 2b,\ s' = \frac{2a\,[b\,(a-b)]^{\frac{1}{2}}}{a-2b}\tan^{-1}\left\{\left(\frac{2\,[b\,(a-b)]^{\frac{1}{2}}}{a-2b}\,z\right)\right\}_{-1}^{+1}$$

$$= \frac{8a\,[b(a-b)]^{\frac{1}{2}}}{a-2b}\sin^{-1}(b/a)^{\frac{1}{2}};$$

$$(2)\ a < 2b,\ s' = \frac{2a\,[b\,(a-b)]^{\frac{1}{2}}}{2b-a}\tan^{-1}\left\{\left(\frac{2\,[b\,(a-b)]^{\frac{1}{2}}}{2b-a}\,z\right)\right\}_{-1}^{+1}$$

$$= \frac{8a\,[b\,(a-b)]^{\frac{1}{2}}}{2b-a}\cos^{-1}(b/a)^{\frac{1}{2}}.$$

[Thus, if $4b = a$, (1) gives $4\sqrt{3}a\,\text{cosec}^{-1}2$, and if $4b = 3a$, (2) gives $4\sqrt{3}a\cos^{-1}\frac{1}{2}\sqrt{3}$, which coincide as they ought].

3. If $\text{OS} = r'$, and p' be the perpendicular from O upon the tangent at S, we have $r' = a^2/r$, $p'/r' = p/r$, or $p' = a^2p/r^2$; whence, if ρ' be the radius of curvature at S of the locus of S,

$$\rho' = r'\frac{dr'}{dp'} = -\frac{a^4}{r^3} + a^2\frac{[r\,(dp/dr) - 2p]}{r^3} = \frac{a^2}{2p - r\,(dp/dr)}, \text{ and } \rho = r\frac{dr}{dp},$$

$$\text{or } \frac{\rho'}{\rho} = \frac{a^2}{2pr\,(dr/dp) - r^2}; \text{ and } \frac{1}{p}\frac{dp}{dr} = \frac{r}{r^2-a^2}, \text{ or } pr\frac{dr}{dp} = r^2 - a^2,$$

$$\text{and } \frac{\rho'}{\rho} = \frac{a^2}{2r^2-2a^2-r^2} = \frac{a^2}{r^2-2a^2} = \frac{OS}{OP-2OS} = \frac{OS}{SP-OS}.$$

4. When there is an inflexion at S, $\rho' = \infty$, which happens when $r^2 = 2a^2$, or $OP = a\sqrt{2}$, and OS is therefore equal to $a/\sqrt{2}$. The circle of curvature at P is then the inverse of the tangent at S, and must pass through O. [The circle of curvature will pass through the origin when $2p\,(dr/dp) = r$, or when $2pr\,(dr/dp) = r^2$; *i.e.*, $2\,(r^2-a^2) = r^2$, or $r^2 = 2a^2$.]

[The equiangular spiral is one of the class of curves which have the property that the chord of curvature through the pole is a constant multiple of the radius vector. Any one of these curves can be drawn to have four-pointic contact with a given curve at any point, and, if the curve be such that the pole of the osculating spiral (S) lie always on the radius vector, the locus of S will be the inverse of the given curve with respect to the origin. If the chord of curvature through the pole be kr, it will be found that $d\rho/ds = (k-1)\cot OPY$ (when S lies on OP), whence

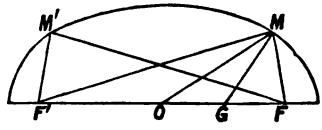
$$r\frac{dr}{dp} \propto p^{k-1}, \quad r^2 = a^{2-k}p^k \pm b^2$$

will be the equation for a curve such that the locus of the pole of the osculating spiral is the inverse, with respect to the circle $r^2 = \pm b^2$, of the curve. If the constant term in this equation be omitted, it is the equation between r and p for the spiral. Perhaps the equation would be better, written in the form $r^2 = a^2(p/c)^k + b^2$, since $k = 2$ (the equiangular spiral) would give $r^2 = p^2 + b^2$, or the involute of a circle which is only a particular case. If we put $k = -2$, the osculating curve is the rectangular hyperbola, and the curve is a central conic. That is, if the rectangular hyperbola of four-point contact be drawn at each point of a central conic, the locus of its *centre* (the pole in this case) is the inverse of the conic with respect to the director circle. No doubt, other particular cases may be readily investigated, such as the Lemniscate of Bernoulli, Cardioid, Parabola, &c.]

10415. (Professor Steiner.)—Soient M, M′, les extrémités de deux diamètres conjugués OM, OM′ d'une ellipse. Démontrer que la somme $\tan^2 \frac{1}{2}FMF' + \tan^2 \frac{1}{2}FM'F'$ est constante.

Solution by Professors Anderson, M.A.; Zerr, M.A.; *and others.*

Denote FM, F′M, FM′, F′M′ by r_1, r_2, r_1', r_2'; then, if s denote half the sum of the sides of the triangle FMF′, or of the triangle FM′F′,



$$\tan^2 \tfrac{1}{2}FMF' + \tan^2 \tfrac{1}{2}FM'F'$$
$$= \{(s-r_1)(s-r_2) + (s-r_1')(s-r_2')\} / \{s(s-2c)\};$$

also, $$s = a + c, \quad r_1 r_2 + r_1' r_2' = a^2 + b^2;$$

therefore $$\tan^2 \tfrac{1}{2}FMF' + \tan^2 \tfrac{1}{2}FM'F' = c^2/b^2.$$

10569. (Professor TARRY.) — Le rapport anharmonique de quatre points A, B, C, D d'une conique est égal à la racine carrée du rapport anharmonique du faisceau P (ABCD) qui a pour centre le pôle P de la droite AB.

Solution by J. C. ST. CLAIR; Professor CHAKRIVARTI; *and others.*

Let PC, PD meet the conic again in C′, D′, and AB in E, G; then CD′, DC′ intersect in a point F lying on AB; and since

$$C'(ABCD) = D'(ABCD),$$

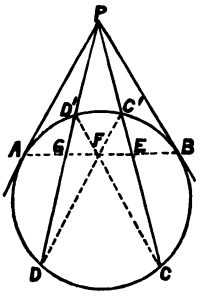
$$\therefore \quad (ABCD) = (ABEF) = (ABFG);$$

and, if $\dfrac{AC}{BC} : \dfrac{AD}{BD} = \lambda,$

we have $\lambda = \dfrac{AE}{BE} : \dfrac{AF}{BF} = \dfrac{AF}{BF} : \dfrac{AG}{GB},$

therefore

$$\lambda^2 = \frac{AE}{BE} : \frac{AG}{BG} = (ABEG) = P(ABCD).$$



[If we project A, B into the circular points at infinity, and use LAGUERRE's theorem for the projection of angles, the theorem follows from Euc. III., 20.]

10416. (Professor G. B. MATHEWS.)—Prove that (1) the conditions that $10x^3 - 8g_2x - 5g_3 = 0$, $315x^4 - 231g_2x^2 + 90g_3x + 25g_2^2 = 0$, may be satisfied by the same value of x, is

$$t^3 - 2 \cdot 5^3 \cdot 79t - 5^3 \cdot 11^3 = 0, \text{ where } t = 27g_2^3/(g_2^3 - 27g_3^2);$$

and (2) that $t = 5^3 \cdot 79 \pm 2^2 \cdot 5 \cdot 13 \cdot 17\sqrt{5} = \pm 5\sqrt{5}\,(4 \mp \sqrt{5})^3\,(\sqrt{5} \pm 2)^4$, where the upper or lower sign is to be taken throughout; and if we put $t = 4\,(1 - y^2)^3/y^4$, one value of y is $\frac{1}{16}\,(\sqrt{5} - 1)^3$.

Solution by H. J. WOODALL; Professor ZERR; *and others.*

The equations given are homogeneous, which greatly simplifies the work. After about seven or eight of the usual steps in ordinary algebraic eliminations, we obtain

$$x = (10395g_3^2 - 4180g_2^3)/(23868g_2g_3);$$

and substituting in the simplest, and reducing, we have

$$11^2\left\{7016625g_3^4 - 2922048g_2^3g_3^2 + 40432g_2^6\right\} + 189893808g_2^3g_3^2 = 0.$$

Substituting $t = 27g_3^3/(g_3^3 - 27g_2^3)$ or $27g_3^2 = (tg_2^3 - 27g_2^3)/t$,

we get $$t^3 - t \times 2 \times 5^3 \times 79 - 5^3 \cdot 11^3 = 0;$$

whence $t = 5^3 \times 79 \pm 5\sqrt{5} \cdot 2^2 \cdot 13 \cdot 17 = \pm 5\sqrt{5}\,(4 \mp \sqrt{5})^3\,(\sqrt{5} \pm 2)^4$,

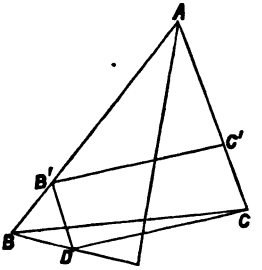
where the upper or lower sign is to be taken throughout.

In the formula $t = 4(1-y^2)^3/y^4$, writing $y = \frac{1}{\sqrt{5}}(\sqrt{5}-1)^3 = \frac{1}{2}(\sqrt{5}-2)$, we obtain $t = 5\sqrt{5}(4-\sqrt{5})^3(\sqrt{5}+2)^4$; whence, if $t = 4(1-y^2)^3/y^4$, we get, for one value, $y = \frac{1}{\sqrt{5}}(\sqrt{5}-1)^3$.

10472. (L. Bénézech.) — Dans un triangle donné ABC, mener, parallèlement à une direction donnée, une sécante B'C' telle que BB' = CC', B' et C' étant les points de rencontre de cette sécante avec AB et AC.

Solution by Professor Anderson, M.A.; J. C. St. Claie; *and others.*

Through C draw a straight line parallel to the given direction, meeting in D the perpendicular from B to the bisector of the angle A; and through D draw a line parallel to CA, meeting AB in B'; then the required secant passes through B', since the triangle BB'D is evidently isosceles, and DB' equal to CC'. If BD be drawn perpendicular to the external bisector of the angle A, another secant will be obtained satisfying the given condition.



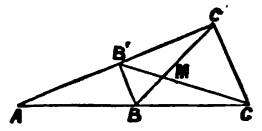
10568. (Professor de Longchamps.)—Soient A, B, C trois points en ligne droite. Par A, on fait passer une droite mobile Δ, et des points B, C on abaisse sur Δ des perpendiculaires BB', CC'. Trouver le lieu décrit par le point de concours des diagonales du trapèze ainsi formé.

Solution by Professor Ignacio Beyens.

Soit M le point de concours des diagonales du trapèze (BB'CC'), on aura

$$\frac{MB}{BC'} = \frac{BB'}{BB'+CC'} = \frac{AB}{AB+AC} = \text{constante},$$

mais le point B est fixe. C' décrit une circonférence ayant (AC) pour diamètre; donc le point (M) qui divise la droite BC' en une rapport constante décrit une circonférence facile à déterminer.

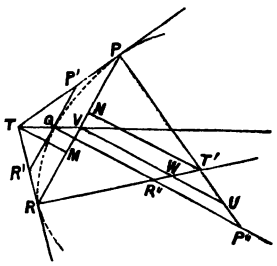


10106. (Professor Stamadas Mukhopâdhyây, B.A.)—P, Q, R are three points on a parabola, PR being parallel to the tangent at Q; r is the radius of a circle touching the tangents at P, Q, R, and r' is the

radius of a corresponding circle touching the normals at P, Q, R. Prove (1) that $\tan\delta = r'/3r$, where δ is the angle between the normal and diameter at Q; and hence (2) deduce for any curve TRANSON'S formula $\tan\delta = \rho'/3\rho$, where δ is the angle of aberrancy at a point (A) of the curve, and ρ, ρ' the radii of curvature of the curve at the point (A), and of the evolute at the point corresponding to (A).

Solution by Professors SWAMINATHA AIYAR, BEYENS, *and others.*

T'P'R' and T'P''R'' are two triangles formed by the tangents and the normals at P, Q, R. TQV is the diameter through Q. T'N, TM, VWU are all parallel to the normal at Q. Now, since perpendiculars from R, P on PT', RT' meet on T'N, it follows PN = RM or VN = VM. Hence T'W = 2WR'',



$$\tan\delta = \tan \mathrm{VTM} = \frac{\mathrm{VM}}{\mathrm{TM}} = \frac{\mathrm{PM}-\mathrm{MR}}{2\mathrm{TM}}$$

$$= \frac{\mathrm{WU}}{\mathrm{P\,R}} = \frac{\frac{1}{2}\mathrm{R''P''}}{2\mathrm{R'P'}} = \frac{\mathrm{R''P''}}{3\mathrm{R'P'}} = \frac{r'}{3r}.$$

TRANSON'S formula follows immediately from this, knowing that the angle of aberrancy is determined by four consecutive points.

10229. (Professor STAMMER.)—**La perpendiculaire élevée au milieu du côté BC d'un triangle rencontre les deux autres côtés AB, AC en des points B' et C' tels que les segments BB' et CC' sont vus de tout point de la circonférence ABC sous des angles égaux ou supplémentaires. Autrement dit, si B' et C' sont deux points divisant un diamètre d'une circonférence en parties harmoniques, et si BC est une corde quelconque perpendiculaire à ce diamètre, les droites BB', CC' sont vues de tout point de la circonférence sous des angles égaux ou supplémentaires.**

Solution by J. J. BARNIVILLE; MORGAN BRIERLEY; *and others.*

Let the origin be the centre of the circle ABC of radius a; let B' be (C, 0), then C' is $(a^2/c, 0)$. Let B and C be $(a\cos\phi, \pm a\sin\phi)$; then BC is $x - a\cos\phi = 0$. Then, if P be any point on the circle, and its coordinates $a\cos\theta$, $b\sin\theta$, we have

$$\tan \mathrm{BPB'} = \frac{a\sin(\theta-\phi) - c(\sin\theta-\sin\phi)}{a\cos(\theta+\phi) - c(\cos\theta-\cos\phi) + a} = \tan \mathrm{CPC'}.$$

10559. (Professor LAMPE, LL.D.)—Let A be the upper end, F the foot, of a vertical staff $AF = l$, standing on the horizontal plane at F under the latitude ϕ; then, if the declination of the sun is supposed to be constant during a day, the shadow of A is known to describe in our latitudes one branch of an hyperbola the declination being $+\delta$, the other branch if the declination is $-\delta$. The asymptotes of all these hyperbolas for different days envelope a parabola tangent at its vertex to the straight line described by the shadow of A on the days of equinox, and having the latus rectum $4l\,tg\,\phi$. The locus of the extremities of the conjugate axes of all hyperbolas is again a parabola with the latus rectum $l\,tg\,\phi$. [These theorems hold, similarly, for the sections of a series of coaxial and concentric cones of revolution by an arbitrary plane.]

Solution by Professor ANDERSON, M.A.

The line AS which joins the upper end of the staff to the centre of the sun's disc describes on any day a right circular cone whose semi-vertical angle is $90° - \delta$, and axis the polar axis of the earth, the declination being supposed positive. When the declination is negative and equal to δ, the same cone will be described again by SA produced through the point A. The curve traced out will be the intersection of this cone with the horizontal plane through F, and will therefore be a conic section. It will be an ellipse, an hyperbola, or a parabola, according as $\phi >$, $<$, or $= 90° - \delta$.

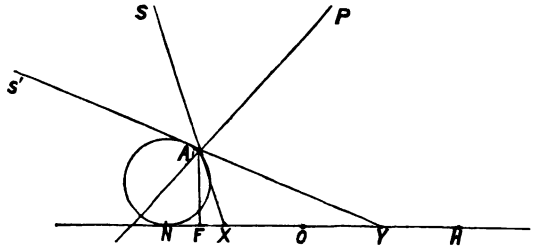


Fig. 1.

At all places between the Arctic and Antarctic Circles, since $90° - \delta$ cannot be $< 66\frac{1}{2}°$, the curve is an hyperbola, one branch being described when the declination is δ, and the other when it is $-\delta$. At the equinoxes δ vanishes, the cone becomes a pair of planes, and the curve a pair of coincident lines, one being described at one equinox, and the other at the other.

In Fig. 1, the plane of the paper is the meridian plane through the vertical staff; AP is the polar axis; AS, AS′ lines through the centre of the sun's disc when in the meridian plane, the declinations being δ and $-\delta$ respectively. Through F draw FX perpendicular to AF, meeting SA and S′A in X and Y the vertices of the hyperbola. Draw the ex-circle of the triangle AXY touching AX. It will touch XY in a focus, N, of the hyperbola, the other focus being H, where YH = XN, and the centre O the middle point of XY.

$$\angle AXF = 90° - (\phi - \delta), \quad \angle AYF = 90° - (\phi + \delta), \quad \text{and} \quad AF = l;$$

therefore $XY = l\{\tan(\phi+\delta)-\tan(\phi-\delta)\} = \dfrac{2l\tan\delta\sec^2\phi}{1-\tan^2\phi\tan^2\delta}$.

Again, YN is equal to half the sum of the sides of the triangle AXY

$$= \tfrac{1}{2}l\{\sec(\phi-\delta)+\sec(\phi+\delta)+\tan(\phi+\delta)-\tan(\phi-\delta)\},$$

and $$ON = YN-\tfrac{1}{2}XY = \tfrac{1}{2}l\{\sec(\phi-\delta)+\sec(\phi+\delta)\}$$

$$= \frac{l\sec\phi\sec\delta}{1-\tan^2\phi\tan^2\delta}.$$

Hence the semi-axes, major and minor (a, b), of the hyperbola are

$$a = \frac{l\tan\delta\sec^2\phi}{1-\tan^2\phi\tan^2\delta}, \quad b = (ON^2-OX^2)^{\frac{1}{2}} = \frac{l\sec\phi}{(1-\tan^2\phi\tan^2\delta)^{\frac{1}{2}}}.$$

Also, $$FO = FX+XO = \frac{l\tan\phi\sec^2\delta}{1-\tan^2\phi\tan^2\delta}.$$

In Fig. 2, the plane of the paper is the horizontal plane through F the foot of the staff; OK, OK′ are a pair of asymptotes, the points B, B′

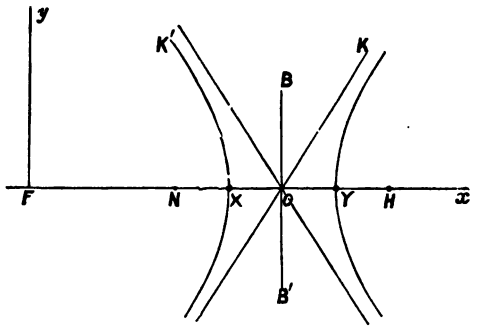


Fig. 2.

being the extremities of the axis-minor. Taking FO and a perpendicular to it through F as axes of coordinates, the equations of the asymptotes are

$$x = \frac{l\tan\phi\sec^2\delta}{1-\tan^2\phi\tan^2\delta} \pm y\tan\delta\sec\phi\,(1-\tan^2\phi\sec^2\delta)^{\frac{1}{2}},$$

At the equinoxes these lines coincide, and we get the straight line $x = l\tan\phi$. Transferring the origin to the point where this straight line cuts the axis of x, the equations of the asymptotes become

$$x = l\tan\phi\,\frac{\tan^2\delta\sec^2\phi}{1-\tan^2\phi\tan^2\delta} \pm y\tan\delta\sec\phi\,(1-\tan^2\phi\sec^2\delta)^{\frac{1}{2}};$$

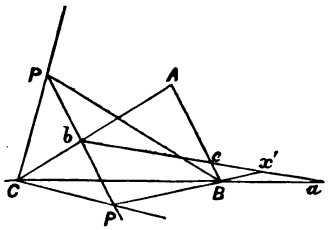
hence $$\pm y = \frac{x}{\tan\delta\sec\phi\,(1-\tan^2\phi\tan^2\delta)^{\frac{1}{2}}}$$

$$-l\tan\phi\tan\delta\sec\phi\,(1-\tan^2\phi\tan^2\delta)^{\frac{1}{2}}.$$

If we take the positive direction of x in the direction OF, this equation is of the form $y = mx + a/m$, and therefore the asymptotes envelope a parabola whose vertex is at the origin and latus rectum $= 4l \tan \phi$. The equation of the locus of B referred to the new origin, and with the new direction of the axis of x, is easily found to be $y^2 = -l \tan \phi x + l^2$. Its latus rectum is, therefore, $l \tan \phi$.

4014. (C. H. Hinton.)—Let ABC be a triangle, and through a on BC draw a straight line cutting AC in b, AB in c. From centre b distance bC, draw a circle; from centre c distance cB, draw a circle cutting the other in P, P′. Bisect the external or internal angle of the triangle Pbc or $P'bc$ by a straight line cutting ab in x. Let x' be a point on another straight line $ab'c'$ similarly obtained, and so on; then x, x', x''... line on a straight line passing through A.

Solution by Professors Anderson, Décamps, *and others.*



The problem is identical with finding the locus of the points X and X′, where the line bC which passes through a is divided internally and externally in the ratio of Cb to cB, a being a fixed point on CB.

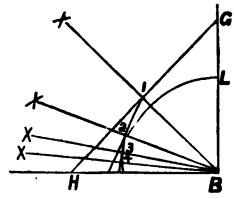
The following construction, which is easily proved, determines the points x and x'. Draw CP making an angle PCA with CA equal to $\frac{1}{2}$A, and CP′ perpendicular to CP. Through b draw PP′ parallel to AB, and join PB and P′B. These lines meet ab in the required points.

Thus x is the vertex of the triangle Pbx whose base Pb is parallel to AB, whose sides Px and bx pass through fixed points B and a, and whose base angles lie on two fixed straight lines CA, CB intersecting in a point lying on the line aB. The point x' is, in like manner, the vertex of the triangle $bP'x'$, which fulfils similar conditions.

Taking the triangle ABC as triangle of reference, and supposing the areal coordinates of a to be 0, β, γ, the locus is found, without difficulty, to be the lines $c\gamma y + b\beta z = 0$, $c\gamma y - b\beta z = 0$.

10082. (D. Biddle.)—The circumference of a circle being represented as to length, by a given straight line; show how (by *pure* geometry) to find the radius, with at least sufficient accuracy for any practical purpose.

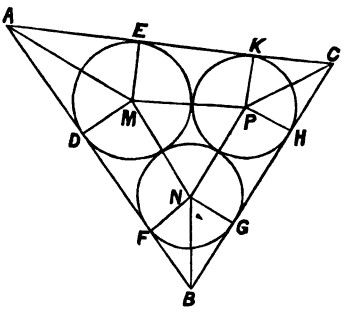
Solution by the PROPOSER; Rev. J. L. KITCHIN; *and others.*



The radius may be found to any degree of accuracy by an extension of the following method:—Let BG = ¼ circumference. Draw BH at right angles, and equal to BG. Join GH and bisect in 1. On BH mark a point equidistant with 1 from B; join and bisect in 2, and so on. It is clear that the points 1, 2, 3, 4, &c., vary in distance from BH as the size of the angles denoted; the angles are the same aliquot parts of the quadrant as the several distances are of BG; and the ratio of each such distance (from BH) to the distance from B, is that of the sine of the angle to its radius. When the process is carried *ad infinitum*, $d\theta = \sin d\theta$. Accordingly, the series tends towards a point on BH which marks the exact position of the circumference (= 4BG), having B for its centre. Each successive point of the series approximates, in its distance from B, more closely than its predecessor to the required length, and the eighth exceeds only by ·0000023. [Other constructions were given by Messrs. KITCHIN, BARNIVILLE, WOODALL, and many others, for which we regret that we cannot anyhow find space.]

4219. (N'IMPORTE.)—Given the radii of three circles inscribed in a triangle, so that each circle touches the other two, and two sides of the triangle; find the sides of the triangle.

Solution by Professors ZERR, BEYENS, *and others.*



Let ABC be the triangle; M, N, P the centres of the circles; MD = EM = a, NF = NG = b, PH = PK = c. Let $a+b+c=s$,

$\angle PMN = \delta$, $\angle NMD = y$,

$\angle PME = \rho$, $\angle EMD = w$;

then

$$\tan \tfrac{1}{2}y = (b/a)^{\frac{1}{2}}, \quad \tan \tfrac{1}{2}\rho = (c/a)^{\frac{1}{2}},$$

$$\tan \tfrac{1}{2}\delta = (bc/as)^{\frac{1}{2}},$$

$$A = \pi - w = \pi - 2\pi + (\delta + y + \rho) = (\delta + y + \rho) - \pi.$$

$$\tan \tfrac{1}{2}A = -\cot \tfrac{1}{2}(\delta + y + \rho) = \frac{\tan \frac{1}{2}\delta \tan \frac{1}{2}y + \tan \frac{1}{2}\delta \tan \frac{1}{2}\rho + \tan \frac{1}{2}y \tan \frac{1}{2}\rho - 1}{\tan \frac{1}{2}\delta + \tan \frac{1}{2}y + \tan \frac{1}{2}\rho - \tan \frac{1}{2}\delta \tan \frac{1}{2}y \tan \frac{1}{2}\rho},$$

$$AD = \frac{a}{\tan \frac{1}{2}A} = \frac{a(abc)^{\frac{1}{2}} + a(abs)^{\frac{1}{2}} + a(acs)^{\frac{1}{2}} - bca^{\frac{1}{2}}}{bc^{\frac{1}{2}} + cb^{\frac{1}{2}} + (bcs)^{\frac{1}{2}} - as^{\frac{1}{2}}};$$

similarly, $BF = \dfrac{b(abc)^{\frac{1}{2}} + b(abs)^{\frac{1}{2}} + b(bcs)^{\frac{1}{2}} - ac\,b^{\frac{1}{2}}}{ac^{\frac{1}{2}} + ca^{\frac{1}{2}} + (acs)^{\frac{1}{2}} - bs^{\frac{1}{2}}}$, $DF = 2(ab)^{\frac{1}{2}}$;

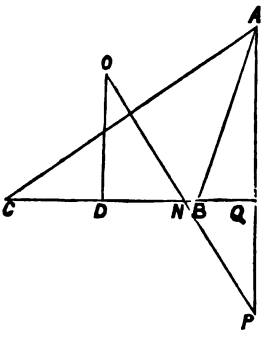
therefore $AB = AD + DF + FB$

$$= \frac{(a+b)^2\left[c(ab)^{\frac{1}{2}} - b(ac)^{\frac{1}{2}} - a(bc)^{\frac{1}{2}} + c\left\{(as)^{\frac{1}{2}} + (bs)^{\frac{1}{2}} + (cs)^{\frac{1}{2}}\right\} - (abcs)^{\frac{1}{2}} + ss\right]}{\left\{ac^{\frac{1}{2}} + ca^{\frac{1}{2}} + (acs)^{\frac{1}{2}} - bs^{\frac{1}{2}}\right\}\left\{bc^{\frac{1}{2}} + cb^{\frac{1}{2}} + (bcs)^{\frac{1}{2}} - as^{\frac{1}{2}}\right\}}.$$

Similar expressions follow for BC and CA.

10600. (Professor Hudson, M.A.)—If the mid-centre of a triangle lies on a side of the triangle, prove that the difference between the angles to which that side is adjacent is a right angle. [The mid-centre is the centre of the mid-circle, that is to say, of the circle that passes through the middle points of the sides.]

Solution by R. H. W. Whapham, B.A.; J. D. Williams; *and others.*



Let P, O, N be ortho-, circum-, and mid-centres, and D the mid-point of BC;

then $OP^2 = 4ON^2 = 4(OD^2 + ND^2)$

$= 4\left\{R^2\cos^2 A + \frac{1}{4}R^2\right\}$;

but $OP^2 = R^2(1 - 8\cos A\cos B\cos C)$,

$\therefore\ 4\cos^2 A + 1 = 1 - 8\cos A\cos B\cos C$;

therefore $\cos(B-C) = 0$.

[The coordinates of the mid-centre are $\cos(B-C)$, $\cos(C-A)$, $\cos(A-B)$;

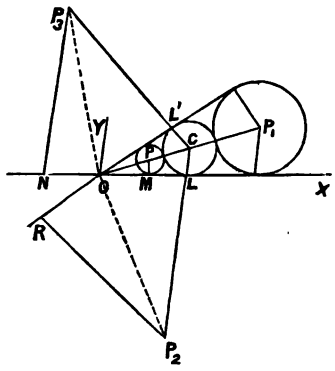
hence, substituting in the equation to BC, we get $\cos(B-C) = 0$.

Geometrically, if PA meet BC at right angles in Q, then PQ is a tangent to the mid-circle, and therefore PA is a tangent to the circumcircle, P being their centre of similarity; hence PAB is equal to the angle in the alternate segment C, and therefore B exceeds C by a right angle.]

10507. (J. White.)—A circle of given radius a rolls upon a fixed straight line, and from a given point in it tangents are drawn to this circle; prove that the loci of the centres of the four circles which can be described to touch the rolling circle, the given line, and the tangents from the given point, are two equal parabolas, and the curve $x^2(y-a)^2 = 4ay^3$.

Solution by D. T. Griffiths; Professor Anderson; *and others.*

Let O be the fixed point in the given line OX. Let C be the centre of the circle which rolls upon OX, and P, P_1, P_2, P_3 the centres of the four circles which touch OL . OL′ and the circle with centre C.



Since O is the centre of similitude of the circles with centres C and P, we have

$$\frac{CL}{PM} = \frac{CP + PO}{PO};$$

$$\therefore \quad \frac{a}{y} = \frac{a + y + (y^2 + x^2)^{\frac{1}{2}}}{(y^2 + x^2)^{\frac{1}{2}}};$$

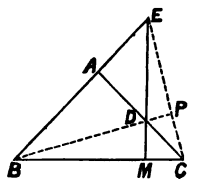
therefore P is a point on the curve $x^2(y-a)^2 = 4ay^3$.

In the same way it may be shown that P_1 is a point on this curve. The points R, P_2, L, O are concyclic; $\therefore \angle RP_2L = L'OL$; $\therefore OP_2L = COL$; $\therefore COP_2 =$ right angle; $\therefore OL^2 = CL \cdot LP_2$; $\therefore$ locus of P_2 is $x^2 = -ay$. In the same way it may be shown that $COP_3 =$ right angle; $\therefore OL'^2 = CL' \cdot P_3L'$; $\therefore ON^2 = aP_3N$; $\therefore$ locus of P_3 is $x^2 = ay$.

10293. (G. Russo.)—En un point quelconque de la base BC d'un triangle isoscèle ABC, on élève une perpendiculaire, qui rencontre AC en D, AB en E. Démontrer que le lieu du point de rencontre des droites BD, CE est une ellipse.

Solution by Rev. T. Galliers, M.A.; J. C. St. Clair; *and others.*

Let A be projected orthogonally on to a plane through BC, so that BAC becomes a right angle. Then D is the orthocentre of $\triangle EBC$, and therefore BPC is a right angle. Hence in this case the locus of P is the circumcircle of $\triangle ABC$. Therefore in the original figure the locus is an ellipse having BC for one axis, and A for an extremity of the other axis.



10264. (Professor Décamps.)—Si, dans un tétraèdre OABC, trirectangle en O, α représente l'angle dièdre formé par les faces ABC et OBC, on a $\tan^2\alpha = \tan^2 ABO + \tan^2 ACO$.

Solution by J. J. Barniville; Rev. T. Galliers, M.A.; *and others.*

Draw AD perpendicular to BC. Then

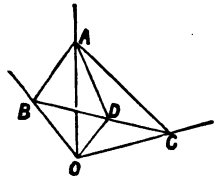
$$AO^2+OB^2=AB^2=BD^2+DO^2+AO^2;$$

therefore $OB^2=OD^2+DB^2$, and OD is perpendicular to BC. Again, $\tan^2 ABO+\tan^2 ACO$

$$=\frac{AO^2}{OB^2}+\frac{AO^2}{OC^2}=AO^2\left(\frac{OB^2+OC^2}{OB^2.OC^2}\right)$$

$$=AO^2\frac{BC^2}{OB^2.OC^2}=AO^2\left(\frac{BC}{OD.BC}\right)^2$$

$$=\frac{AO^2}{OD^2}=\tan^2\alpha.$$



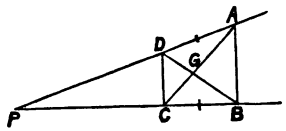
10371. (R. Knowles, B.A.)—The sides AB, CD of a quadrilateral inscribed in a conic, centre O, are parallel; the diagonals AC, BD intersect in G; a second conic, centre O′, is drawn through A, B, C, D; prove that the points O, O′, G are collinear.

Solution by R. H. W. Whapham, B.A.; C. Morgan, M.A.; *and others.*

Take P, where BC and AD meet for origin, and BC, AD for axes.

Let AB be $ax+by-1=0$, and CD, $a'x+b'y-1=0$; therefore, any conic through ABCD being

$$\lambda xy+(ax+by-1)(a'x+b'y-1)=0,$$



the locus of centre is $2aa'x^2-2bb'y^2-(a+a')x+(b+b')y=0$.

But, since AB and CD are parallel, we have

$$\frac{a}{b}=\frac{a'}{b'}=\frac{a+a'}{b+b'}=\frac{(aa')^{\frac{1}{2}}}{(bb')^{\frac{1}{2}}}=k;$$

therefore the centre-locus reduces to

$$(kx-y)\left\{2kbb'x+2bb'y-(b+b')\right\};$$

whence we see that O, O′, G lie on the right line $kx-y=0$, since G is the centre of the two-line conic AC, BD; and P (the origin) is the centre of the two-line conic AD, BC. The line

$$2kbb'x+2bb'y-(b+b')=0,$$

evidently passes through the mid-points of BC and AD, and contains the centre of the two-line conic AB, CD.

10517. (Professor Steiner.)—Par un point fixe P de l'axe focal d'une ellipse ou d'une hyperbole, on mène une sécante quelconque rencontrant la courbe en M, M′. Démontrer qu'il existe une constante r, telle que le produit $(FM+r)(FM'+r)$ a une valeur constante, F étant l'un des foyers.

Solution by Professor ANDERSON, M.A.; H. J. WOODALL; *and others.*

The polar equation of a conic referred to the focus F as origin is $l = \rho(1 - e\cos\theta)$, and that of a straight line through P, $\rho\cos(\theta + \alpha) = d\cos\alpha$, d being the constant distance FP, and α the complement of the angle which the line makes with the major axis. Eliminating θ, we get

$$\rho^2(1 - e^2\sin^2\alpha) - 2\rho(l + ed\cos^2\alpha) + l^2 + 2edl\cos^2\alpha + e^2d^2\cos^2\alpha = 0.$$

The roots of this equation are the distances FM, FM', and hence the quadratic whose roots are FM + r, FM' + r is

$$(\rho - r)^2(1 - e^2\sin^2\alpha) - 2(\rho - r)(l + ed\cos^2\alpha) + l^2 + 2edl\cos^2\alpha + e^2d^2\cos^2\alpha = 0.$$

The product of the roots of this quadratic is

$$r^2 + \{(l + ed)^2 + 2r(l + ed) - (e^2d^2 + 2edl + 2edr)\sin^2\alpha\} / (1 - e^2\sin^2\alpha),$$

which is independent of α, if $e^2(l + ed)^2 + 2re^2(l + ed) = e^2d^2 + 2edl + 2edr$;

hence
$$r = \frac{1}{2} \cdot \frac{e^2d^2 + 2edl - e^2(l + ed)^2}{e^2(l + ed) - ed}.$$

10560. (Professor GENESE, M.A.)—If $\{a_1b_2c_3 \ldots\}$ be a determinant, and $a_\alpha b_\beta c_\gamma \ldots$ a term, prove the following rule for the sign of a term:—Let p be the number of suffixes after α and *less than* α; q, r, &c., the corresponding numbers for β, γ, &c.; then the sign of the term is $(-1)^{p+q+r+\ldots}$.

Solution by H. J. WOODALL; Professor SARKAR; *and others.*

This follows at once because $p + q + r + \ldots$ is identically the number of "inversions" which occur in the term $a_\alpha b_\beta c_\gamma \ldots$.

Thus, for the term $a_5b_1c_4d_3e_2$ of $\{a_1b_2c_3d_4e_5\}$ after the 5, 1, 4, 3 there are 4, 0, 2, 1 less suffixes, 7 in all; and the sign of the term is minus.

Of course, even numbers may be "cast out" from the counting of $p + q + r + \ldots$.

10617. (Rev. H. SEWELL, B.A.)—If a, b, c, ... k be n positive quantities which are not all equal, prove that, (1)

$$(a^m + b^m + c^m + \ldots + k^m)/n > \{(a + b + c + \ldots + k)/n\}^m;$$

and (2) the general theorem of which the above is a particular case, that $\{a^m + b^m + c^m + \ldots + k^m)/n\}^{1/m}$ increases or decreases with m.

Solution by D. BIDDLE; *the* PROPOSER; *and others.*

To prove (2), take δa^p quantities a^q; δb^p quantities b^q, δc^p quantities c^q; ... δk^p quantities k^q; where δ is a common denominator of

all the fractions included in a^p, b^p, c^p, ... k^p, and p, q are positive integers; then

$$\left(\frac{\delta\Sigma a^{p+q}}{\delta\Sigma a^p}\right)^{\delta\Sigma a^p} > (a^q)^{\delta a^p}(b^q)^{\delta b^p}\dots(k^q)^{\delta k^p};$$

therefore $$\left(\frac{\Sigma a^{p+q}}{\Sigma a^p}\right)^{\delta\Sigma a^p} > \left\{a^{pa^p}b^{pb^p}c^{pc^p}\dots k^{pk^p}\right\}^{(q\delta)/p};$$

$$\text{à fortiori} > \left(\frac{\Sigma n^p}{n}\right)^{\Sigma a^p . q\delta/p}, \text{ hence } \left(\frac{\Sigma a^{p+q}}{\Sigma a^p}\right) > \left(\frac{\Sigma a^p}{n}\right)^{q/p};$$

$$\text{i.e., } \frac{\Sigma a^{p+q}}{n} > \left(\frac{\Sigma a^p}{n}\right)^{(p+q)/p} \text{ or } \left(\frac{\Sigma a^{p+q}}{n}\right)^{1/(p+q)} > \left(\frac{\Sigma a^p}{n}\right)^{1/p}.$$

If $p = 1$ and $q = m-1$, the theorem in (1) is at once evident. As a particular case, the following expression decreases as m increases:

$$\left\{\left(\sqrt[m]{a}+\sqrt[m]{b}+\sqrt[m]{c}+\dots+\sqrt[m]{k}\right)/n\right\}^m.$$

10200. (Captaine De Rocquigny.)—Neuf nombres entiers consécutifs sont disposés en carré de la manière suivante:

$n-4$,	$n+2$,	$n-2$
$n-1$,	n,	$n+4$
$n+3$,	$n-3$,	$n+1$

Le centre est occupé par le nombre du milieu; les autres nombres se succèdent, d'après la loi du cavalier, dans l'ordre de grandeur croissante. Choisir n de manière que la valeur de ce *déterminant* du cavalier soit un carré parfait.

Solution by J. J. Barniville; W. J. Greenstreet; *and others.*

The value of the determinant is $28(2n-1)$. In order that this may be a perfect square, we must have $2n-1 = 7p^2$, or $n = \frac{1}{2}(7p^2+1)$; where p is any odd number. A more general formula is $n = 14q^2-14q+4$, where q may have any value.

10256. (R. H. W. Whapham, B.A.)—From any point P on a semicubical parabola there are drawn normals PQ, PQ′ to the parabola; PN is drawn perpendicular to the axis, meeting it in N, and V is the vertex of the semicubical parabola: if PQ. PQ′ meet the axis in G, G′, respectively, show that $VN = 3NG = 3VG'$.

Solution by Rev. T. Galliers, M.A.; Professor Sarkar; *and others.*

Let the equations of the involute and semicubical parabola be

$$y^2 = 4ax, \quad 27ay^2 = 4(x-2a)^3 \dots\dots\dots\dots\dots\dots (1, 2).$$

Let P be (X, Y); then the m's of the normals to (1) through P are given by the cubic in μ, $a\mu^3 - (X - 2a)\mu + Y = 0$ .....(3); let the roots of this equation be m, m', m' [two of them being equal by (2)]. Hence $X - 2a = 3am'^2 = \frac{3}{4}am^2$;

also $AG = am^2 + 2a$, $A = am'^2 + a$, also $AV = 2a$, $AN = X$;

therefore $VN = 3NG = 3VG'$.

10288. (W. J. Greenstreet, M.A.)—Find the envelope of the sides of a triangle which is inscribed in an ellipse, and whose orthocentre is a fixed point.

Solution by Rev. T. Galliers, M.A.; M. Brierley; *and others.*

If (X, Y) be the point, α, β, γ the excentric angles of the corners,

$$(a^2 + b^2)\cos\alpha - (a^2 - b^2)\cos(\alpha + \beta + \gamma) = 2aX - (a^2 + b^2)(\cos\beta + \cos\gamma),$$
$$(a^2 + b^2)\sin\alpha \ldots = \psi.$$

Squaring and adding,

$$(a^2X^2 + b^2Y^2)(p^2 + q^2) - a^4q^2 - b^4p^2 - 2(a^2 + b^2)(paX + qbY) + (a^2 + b^2)^2 = 0 \quad \ldots\ldots(1).$$

Connecting the parameters in

$$pa^{-1}x + qb^{-1}y = 1 \quad \ldots\ldots (2)$$

of one of the sides, since we know that

$$p = \cos\tfrac{1}{2}(\beta + \gamma) / \cos\tfrac{1}{2}(\beta - \gamma); \quad q = \sin\tfrac{1}{2}(\beta + \gamma) / \cos\tfrac{1}{2}(\beta - \gamma).$$

Differentiating (1) and (2),

$$\left\{(a^2X^2 + b^2Y^2 - b^4)p - (a^2 + b^2)aX\right\}dp + \left\{(a^2X^2 + b^2Y^2 - a^4)q - (a^2 + b^2)bY\right\}dq = 0,$$
$$bx \,.\, dp + ay \,.\, dq = 0.$$

Introducing λ, such that

$$(a^2X^2 + b^2Y^2 - b^4)p - (a^2 + b^2)aX = \lambda bx \quad \ldots\ldots(3),$$
$$(a^2X^2 + b^2Y^2 - a^4)q - (a^2 + b^2)bY = \lambda ay \quad \ldots\ldots(4),$$

multiplying (3) and (4) by p and q, and adding, we have, by the aid of (1) and (2), $(a^2 + b^2)(paX + qbY) - (a^2 + b^2)^2 = \lambda ab$.

Substituting this value of λ in (3) and (4), and putting

$$(a^2X^2 + b^2Y^2 - a^4) / (a^2 + b^2) = A, \quad (a^2X^2 + b^2Y^2 - b^4) / (a^2 + b^2) = B,$$

we get $a(B - xX)p - bxYq = a^2X - (a^2 + b^2)x$

$$-ayX \,.\, p + b(A - yY)q = b^2Y - (a^2 + b^2)y \quad \ldots\ldots(5).$$

Solving equations (5) in p and q, and putting

$$V = A.B - A.x.X - B.y.Y,$$

we have
$$\left.\begin{aligned} V.pa^{-1}x &= (A - yY)\,xX - x^2X^2 + a^2x^2 \\ V.qb^{-1}y &= (B - xX)\,yY - y^2Y^2 + b^2y^2 \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots(6).$$

Adding equations (6), and using (2), we obtain

$$AB - 2AxX - 2ByY - 2xyXY + x^2(X^2 - a^2) - y^2(Y^2 - b^2) = 0;$$

and then, putting for A and B their values, we get

$$\left(2xX - \frac{a^2X^2 + b^2Y^2 - b^4}{a^2 + b^2}\right)\left(2yY - \frac{a^2X^2 + b^2Y^2 - a^4}{a^2 + b^2}\right) + x^2(X^2 - a^2) + y^2(Y^2 - b^2) = 2xyXY.$$

[This agrees with the form in which the equation of the locus is given by Professor WOLSTENHOLME, in whose "Problems" it forms Quest. 1266 of the edition of 1878.]

10369. (R. H. W. WHAPHAM, B.A.)—Show that the last term of the equation whose roots are the squares of the differences of the roots of $x^n = 1$ is n^n.

Solution by J. J. BARNIVILLE; Professor MATZ; *and others.*

$$\frac{d(x^n - 1)}{dx} = nx^{n-1}, \quad \frac{nx^{n-1}}{x^n - 1} = \frac{1}{x - a_1} + \frac{1}{x - a_2} + \ldots \frac{1}{x - a_n},$$

where $a_1, \ldots a_n$ are the nth roots of unity; hence

$$na_1^{n-1} = (a_1 - a_2)(a_1 - a_3)\ldots(a_1 - a_n),\ na_2^{n-1} = (a_2 - a_1)(a_2 - a_3)\ldots(a_2 - a_n),\ \ldots$$
$$\ldots na_n^{n-1} = (a_n - a_1)(a_n - a_2)\ldots(a_n - a_{n-1});$$

hence the product of squares of differences of roots

$$= -n^n(a_1 a_2 \ldots a_n)^{n-1} = -n^n.$$

10613. (J. O'BYRNE CROKE, M.A.)—Prove that the area of an ellipse given by the general equation $(a, b, c, f, g, h)(x, y, 1)^2 = 0$ is $\pi\Delta/C^{\frac{3}{2}}$, where Δ is the discriminant of the equation, and $C = d\Delta/dc$.

Solution by the PROPOSER.

Taking the general equation of the ellipse as

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0,$$

we have (SALMON'S *Conic Sections*, § 152, and § 157, Ex. 3), for the

equation transformed to the centre as origin of coordinates, and principal axes of the ellipse as axes, $(a+b-\mathrm{R})\,x^2+(a+b+\mathrm{R})\,y^2=\dfrac{2\Delta}{h^2-ab}$,

where $\mathrm{R}^2=4h^2+(a-b)^2$, $\Delta=abc+2fgh-af^2-bg^2-ch^2$.

Hence, if α, β be semi-axes,

$$\alpha^2=\frac{2\Delta}{(h^2-ab)(a+b-\mathrm{R})},\quad \beta^2=\frac{2\Delta}{(h^2-ab)(a+b+\mathrm{R})};$$

therefore
$$\alpha^2\beta^2=\frac{4\Delta^2}{(h^2-ab)^2\left\{(a+b)^2-\mathrm{R}^2\right\}}.$$

But
$$\mathrm{R}^2=4h^2+(a-b)^2,$$

$\therefore\quad (a+b)^2-\mathrm{R}^2=(a+b)^2-(a-b)^2-4h^2=4ab-4h^2=4\,(ab-h^2)$;

therefore
$$\alpha^2\beta^2=\frac{4\Delta^2}{(h^2-ab)^2\,4\,(ab-h^2)}=\frac{\Delta^2}{(ab-h^2)^3};$$

therefore
$$\alpha\beta=\frac{\Delta}{(ab-h^2)^{\frac{3}{2}}}=\frac{\Delta}{\mathrm{C}^{\frac{3}{2}}};$$

therefore
$$\text{Area}=\pi\alpha\beta=\pi\Delta/\mathrm{C}^{\frac{3}{2}}.$$

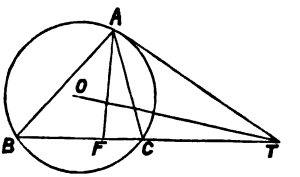
10471. (R. Knowles, B.A.)—In any triangle ABC, prove that (1) the tangent to the circumcircle at A, the symmedian from a, and the sides AB, AC from a harmonic pencil; (2) the tangent, the side BC, and the perpendicular from the circumcentre on the symmedian are concurrent.

Solution by Prof. Anderson, M.A.; R. W. Whapham, B.A.; *and others.*

Let the symmedian from A cut the base in F; then it is easily shown that

$$\mathrm{BF}:\mathrm{FC}=c^2:b^2=\mathrm{BT}:\mathrm{CT};$$

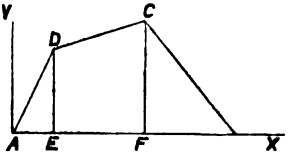
thus A[BFCT] is a harmonic pencil; hence, also, AF is the polar of the point T, and therefore the straight line joining O and T is perpendicular to AF.



3865. (Artemas Martin, LL.D.)—The boundary lines of a quadrangular county ABCD are AB = a, BC = b, CD = c, DA = d, and the diagonal AC = e; find the most equitable position for the county seat, supposing the county uniformly populated.

Solution by Professor G. B. M. ZERR.

Referred to AX, AY as axes, let $(x_1, y_1) \ldots (x_4, y_4)$ be the rectangular coordinates of A, D, C, B; and let μ, ν be the referred areas of the triangles ADC, ABC; then the coordinates of the centroid will be



$$\bar{x} = \frac{\mu(x_1+x_2+x_3)+\nu(x_1+x_3+x_4)}{3(\mu+\nu)},$$

$$\bar{y} = \frac{\mu(y_1+y_2+y_3)+\nu(y_1+y_3+y_4)}{3(\mu+\nu)},$$

$x_1 = 0 = y$; $x_4 = a$; $y_4 = 0$;

$$x_2 = \frac{1}{4ae^2}\{(a^2+e^2-b^2)(d^2+e^2-c^2) - [(c+d+e)(c+d-e)(c+e-d)(d+e-c)(a+b+e)(a+b-e)(a+e-b)(b+e-a)]^{\frac{1}{2}}\},$$

$$y_2 = \frac{1}{4ae^2}\{[(c+d+e)(c+d-e)(c+e-d)(d+e-c)]^{\frac{1}{2}}(a^2+e^2-b^2) + [(a+b+e)(a+b-e)(a+e-b)(b+e-a)]^{\frac{1}{2}}(d^2+e^2-c^2)\},$$

$$x_3 = \frac{a^2+e^2-b^2}{2a}, \quad y_3 = \frac{[(a+b+e)(a+b-e)(a+e-b)(b+e-a)]^{\frac{1}{2}}}{2a}.$$

Now μ and ν are known; hence, by substituting, we get the value of $\bar{x}$, y in terms of a, b, c, d, e, and the centroid is known. This gives the most equitable position.

10572. (R. TUCKER, M.A.)—Tangents to a parabola at the points P, Q, R, form a tangent-triangle pqr, whose sides make angles θ_1, θ_2, θ_3 with the axis of the curve. Show that (1) the rectangular coordinates of the "S"-point of pqr are given by

$$X = \frac{2a}{D}\left\{\sin(\theta_1+\theta_2+\theta_3) + \Sigma\sin(\theta_1+\theta_2-\theta_3) - \frac{\Sigma\sin(\theta_1+\theta_2-\theta_3)+\Sigma\sin(\theta_1+\theta_2-3\theta_3)}{2D'}\right\},$$

$$Y = \frac{a}{D}\left\{\frac{\Sigma\cos(\theta_1+\theta_2-\theta_3)-\Sigma\cos(\theta_1+\theta_2-3\theta_3)}{D'} - 4\cos(\theta_1+\theta_2+\theta_3)\right\},$$

where $D \equiv 8\sin\theta_1\sin\theta_2\sin\theta_3$, $D' \equiv 1-\cos(\theta_1-\theta_2)\cos(\theta_2-\theta_3)\cos(\theta_3-\theta_1)$; (2) if two of the tangents are drawn at the extremities of the L.R., then the locus of the "S"-points is

$$Y^2(2X+3a)+(2X+a)(X+a)^2 = 0;$$

and (3) if the angles made by the tangents are θ, $\frac{1}{2}\pi+\theta$, and $\frac{1}{2}\pi-2\theta$, then the locus is $4\lambda(4-\lambda)y^2 = a^2[\lambda^2-\lambda+4]^2$, where $2X+3a = a\lambda$.

Solution by the PROPOSER.

The tangents to the parabola are given by equations of the form

$$y\cos\theta_1 - x\sin\theta_1 - a\operatorname{cosec}\theta_1\cos^2\theta_1 = 0 \quad \text{.....(i.)};$$

and the sides (a', b', c') are $qr = 8\mathrm{D}\sin(\theta_2 \sim \theta_3)$, $rp = \ldots$, $pq = \ldots$. Then, $\mathrm{Y}\cos\theta_1 - \mathrm{X}\sin\theta_1 - a\operatorname{cosec}\theta_1\cos^2\theta_1 =$ perpendicular from "S"-point on qr

$$= 2a'\Delta/(a'^2 + b'^2 + c'^2) = 4\mathrm{D}\sin(\theta_1 - \theta_2)\sin(\theta_2 - \theta_3)\sin(\theta_3 - \theta_1)\sin(\theta_2 - \theta_3)/\mathrm{D}'$$

$$= \mathrm{D}''\sin(\theta_2 - \theta_3) \quad \text{.....(ii.)},$$

$$\mathrm{Y}\cos\theta_2 - \mathrm{X}\sin\theta_2 - a\operatorname{cosec}\theta_2\cos^2\theta_2 = \mathrm{D}''\sin(\theta_3 - \theta_1) \quad \text{.....(iii.)}.$$

From (ii.) and (iii.) we get, by eliminating X, Y in succession, the values given.

(i.) In the example, $\theta_1 = \theta$, $\theta_2 = 45°$, $\theta_3 = 135°$; therefore

$$\frac{2(\mathrm{X}+a)}{a} = -\cos 2\theta, \quad \frac{\mathrm{Y}}{a} = \frac{\cos 2\theta\cos\theta}{2\sin\theta}, \quad i.e., \quad \frac{\cos\theta}{-\mathrm{Y}} = \frac{\sin\theta}{\mathrm{X}+a},$$

and therefore $$\frac{1}{\mathrm{Y}^2 + (\mathrm{X}+a)^2} = -2\frac{\mathrm{X}+a}{a}\Big/\left[\mathrm{Y}^2 - (\mathrm{X}+a)^2\right];$$

whence $$\mathrm{Y}^2\left[2\mathrm{X}+3a\right] + (2\mathrm{X}+a)(\mathrm{X}+a)^2 = 0.$$

(ii.) $$\mathrm{D} = 4\sin 2\theta\cos 2\theta, \quad \mathrm{D}' = 1;$$

$$\mathrm{X} = a\cos 6\theta/2\cos 2\theta, \quad \mathrm{Y} = a(\cos^2 4\theta + 3\cos^2 2\theta)/2\sin 2\theta\cos 2\theta;$$

whence, eliminating 2θ, we get the given result.

If the tangent triangle is equilateral, the value of X is $-a$, as it should be; for then the "S"-point coincides with the orthocentre of a tangent-triangle.

If the tangent-triangle is formed by tangents at the extremities of a focal chord and a tangent parallel to the chord, then $\mathrm{X} = -a/2$, which is evident from the geometry of the figure.

10651. (Prince CAMILLE DE POLIGNAC.)—A right line of constant length moves so that three of its points remain on three cylindrical surfaces the generators of which are parallel to a given line. Prove that any other point must remain on a cylindrical surface of the same nature.

Solution by the PROPOSER; Professor ANDERSON; *and others.*

This is a generalisation of Quest. 10109, by Professor MANNHEIM, and the solution is immediate. Taking the axis of z parallel to the given line, the given surfaces are

$$\phi_1(x_1y_1) = 0, \quad \phi_2(x_2y_2) = 0, \quad \phi_3(x_3y_3) = 0.$$

Let a, b denote the direction cosines of the moveable line at any moment; l_1, l_2, l_3 the distances from the three given points to an arbitrary point xyz on the line. Then we have the three pairs of equations

$$x - x_{1,2,3} = al_{1,2,3}, \qquad y - y_{1,2,3} = bl_{1,2,3},$$

in all nine equations. Now, owing to the parallelism of the axis of z

with the generators of the cylindrical surfaces, the coordinate z has been abliminated (according to Professor SYLVESTER'S happy terminology). Hence the result of the elimination of the eight parameters a, b, $x_{1,2,3}$, $y_{1,2,3}$ between the nine equations must, from necessity be of the form $\phi(x, y) = 0$. If the three given surfaces are planes, we have Professor MANNHEIM'S theorem.

If ϕ_1, ϕ_2 are planes, then the arbitrary point moves on a cylindrical surface of the same order as ϕ_3. For the above equations show that in this case $x_3 y_3$ can be expressed linearly in terms of x, y.

[If a, b, c be the coordinates of the point whose locus is sought, and l, m, n the direction cosines of the line in any position, we have

$f_1(a + lr_1,\ b + mr_1) = 0,\quad f_2(a + lr_2,\ b + mr_2) = 0,\quad f_3(a + lr_3,\ b + mr_3) = 0,$

the axis of z being the line to which the generators are parallel, and r_1, r_2, r_3 constant lengths. Eliminating l and m between these three equations, we get an equation of the form $F(a, b) = 0$, which denotes a cylindrical surface whose generators are parallel to the axis of z.]

10438. (Colonel SERVAIS).—Si en un point quelconque A d'une conique, on mène un cercle tangent dont le rayon soit double du rayon de courbure en ce point: démontrer que (1) le pôle P, par rapport au cercle, de la corde d'intersection BC de la conique et du cercle est sur la conique; (2) les deux droites BC et PA sont conjuguées harmoniques par rapport à la conique.

Solution by R. KNOWLES, B.A.; Prof. ANDERSON, M.A.; *and others.*

Let the equations to the conic and circle referred to the tangent and normal at A be $x^2 + bxy + cy^2 + ey = 0$, $x^2 + y^2 + 2ey = 0$ ..... (1, 2); the equation to BC is $bx + (c-1)y - e = 0$ ..... (3); whence the coordinates of P are found to be be/c, $-e/c$; and, as these values satisfy (1), $\therefore$ (1) P is on the conic; (2) the equation to PA is $x + by = 0$, and the pole P′ of BC with respect to the conic is on this line; $\therefore$ if PA, BC meet in O, the lines OPP′, OBC will form an harmonic pencil with the tangents from O to either the circle or the conic. [SALMON'S CONICS, 3rd Ed., p. 133.]

10595. (Professor DE LONGCHAMPS.)—On donne a et a_1; puis, on calcule a_2, a_3, ... a_n, par voie récurrente, au moyen de la formule

$$a(n-1)\,a_n = a_1 a_{n-1} + a_2 a_{n-2} + \ldots + a_{n-1} a_1;$$

démontrer que les nombres a_1, a_2, ... a_n forment une progression géométrique.

Solution by W. J. GREENSTREET, M.A.; FABIAN ROSENBERG; *and others.*

We easily get

$$a_2 = a_1^2/a;\quad a_3 = a_1^3/a^2;\quad a_4 = a_1^4/a^3;\quad \ldots\ a_n = a_1^n/a^{n-1},$$

a Geometric Progression.

10621. (Professor SYLVESTER.)—The points of a system containing $2m$ points H and n points O are to be connected together either by single or double bonds, in such a manner that two bonds shall pass through each O, and one through each H. Find the number of distinct species of graphs that can be formed subject to this condition. *Ex.*—Let a, b, c, d and α, β, γ, δ, ϵ be the points H, O; then, denoting any two points x, y with a single bond between them by xy, and with a double bond by $(xy)^2$, we have the following system of conjunction (11 distinct graphs in all):—

$ab,\ cd,\ a\beta,\ \beta\gamma,\ \gamma\delta,\ \delta\epsilon,\ \epsilon a$	$ab,\ ca,\ d\beta,\ a\gamma,\ \beta\gamma,\ (\delta\epsilon)^2$
$ab,\ cd,\ a\beta,\ \beta\gamma,\ \gamma a,\ (\delta\epsilon)^2$	$aa,\ ba,\ c\beta,\ d\beta,\ \gamma\delta,\ \delta\epsilon,\ \epsilon\gamma$
$ab,\ ca,\ da,\ \beta\gamma,\ \gamma\delta,\ \delta\epsilon,\ \epsilon\beta$	$aa,\ ba,\ c\beta,\ d\gamma,\ \gamma\delta,\ \delta\epsilon,\ \epsilon\beta$
$ab,\ ca,\ da,\ (\beta\gamma)^2,\ (\delta\epsilon)^2$	$aa,\ ba,\ c\beta,\ d\gamma,\ \beta\gamma,\ (\delta\epsilon)^2$
$ab,\ ca,\ d\beta,\ a\beta,\ \gamma\delta,\ \delta\epsilon,\ \epsilon\gamma$	$aa,\ b\beta,\ c\gamma,\ d\delta,\ a\beta,\ \gamma\epsilon,\ \epsilon\delta$
$ab,\ ca,\ d\beta,\ a\gamma,\ \gamma\delta,\ \delta\epsilon,\ \epsilon\beta$	

Solution by H. J. WOODALL.

Let $[\frac{1}{2}P]$ denote the greatest integer in $\frac{1}{2}P$; and let such a bracket as (A, B, C, D, ... Z; Ω) stand for the arrangement of the H points among the O points such that 1st, 2nd, ... pairs of H points are connected in a chain with A, B, ... points O; also, $A+B+C+\ldots+Z=n-\Omega$; *i.e.*, there are Ω disengaged points O, and the number of A, B, C, ..., Z is m, but any one or more or none at all of A, B, C, ..., Z may be zero.

Then, examining the given example, we find that the number of graphs connected with one such arrangement given above is equal to $[\frac{1}{2}\Omega]$. Hence the total number will be equal to the sum $\Sigma\,[\frac{1}{2}\Omega]$, which we proceed to show how to find.

There are, of A, B, C, D, ..., Z, m letters. Consider a case where there are $p=$ A, $q=$ B, $r=$ C, &c.; then there will be $\frac{n!}{p!\,q!\,r!}$ ... different "arrangements" with same numerical values, only differently arranged, which will only count as one. *E.g.*, aa, ba, $c\beta$, $d\beta$, &c., and $a\beta$, $b\beta$, ca, da, &c., count only as one "arrangement."

Values.					No. of Graphs.
A	B	C	D	...	$[\frac{1}{2}n]+[\frac{1}{2}(n-1)]+\ldots+[\frac{1}{2}]+1=\Sigma_2^n\{[\frac{1}{2}r]\}+1$
0	0	0	0	...	$=\frac{1}{4}(n-1)(n+2)-[\frac{1}{2}(n-1)]+1,$
1	1	0	0	...	$[\frac{1}{2}(n-2)]+[\frac{1}{2}(n-3)]+\ldots+[\frac{2}{2}]+1=\Sigma_2^{\cdot}\ \{[\frac{1}{2}r]\}+1$
					$=\frac{1}{4}[(n-3)](n)-[\frac{1}{2}(n-3)]+1,$
					for any value of A and B. Number of graphs is $\Sigma_1\{\frac{1}{4}[(n-r)(n-r+3)]+[\frac{1}{2}(n-r)]+1\}$.

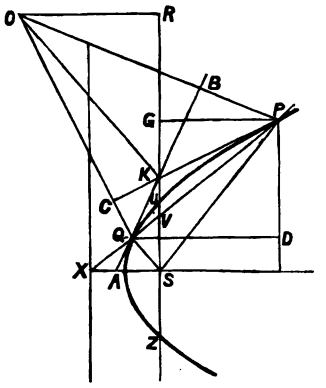
The upper limit depends upon whether n is even or odd; if $n=2k$, the upper limit is $k-1$; if $n=2k+1$, the upper limit is $k-1$; in fact the limit is $[\frac{1}{2}(n-2)]$.

This summation may be effected by means of Algebra, or perhaps by the Calculus of Finite Differences, and then the summation can be extended to variations in the value of A, B, and also C, and so on.

10646. (R. TUCKER, M.A.)—If tangents KP, KQ are drawn to a parabola from any point K on the latus rectum, prove geometrically that the orthocentre of the triangle PKQ lies on a straight line parallel to the directrix.

Solution by Professors ANDERSON, M.A.; ZERR, M.A.; *and others.*

Draw PO and QO perpendicular to KQ and KP respectively, and from O their point of intersection, which is the orthocentre of the triangle KPQ, draw OR perpendicular to SK. It is clear that PQ passes through X the foot of the directrix, and that [KUVZ] is an harmonic range. Also, K is equidistant from PG and QD, lines drawn through P and Q parallel to the axis of the parabola. We have



$$OK : PQ = KB : BP = SV : QV = SG : GP = 2SK : QD;$$

also, $PQ : PD = OK : OR$;

$$\therefore OR = 2SK \cdot PD/QD = 2SK \cdot SV/SX = 2SU^2/SX = 2SU = \text{lat. rect. of parabola.}$$

Hence the locus of O is a straight line parallel to the directrix, whose distance from the focus is equal to the latus rectum.

[The PROPOSER sent a solution by Euclidian geometry only, for which, however, we regret that we could not find room.]

10682. (Professor CAYLEY, F.R.S.)—If ω be an imaginary cube root of unity, and

$$y = \frac{(\omega - \omega^2)\,x + \omega^2 x^3}{1 - \omega^2(\omega - \omega^2)\,x^2},$$

prove that

$$\frac{dy}{(1-y^2)^{\frac{1}{2}}(1+\omega y^2)^{\frac{1}{2}}} = \frac{(\omega-\omega^2)\,dx}{(1-x^2)^{\frac{1}{2}}(1+\omega x^2)^{\frac{1}{2}}}.$$

Solution by J. D. H. DICKSON, M.A.; Prof. SIRCOM, M.A.; *and others.*

For shortness write $\omega - \omega^2 = \lambda$, and $\omega^2 x^2 = \xi$; then we find easily

$$dy/\lambda dx = (1-\xi^2)/(1-\lambda\xi)^2.$$

Now $1 - y^2 = \dfrac{(1-\lambda\xi)^2 - (\lambda x + \xi x)^2}{(1-\lambda\xi)^2} = \dfrac{(1+\lambda x - \lambda\xi + \xi x)(1 - \lambda x - \lambda\xi - \xi x)}{(1-\lambda\xi)^2}$.

Here, $1 + \lambda x - \lambda\xi + \xi x = 1 + \omega x - \omega^2 x - x^2 + \omega x^2 + \omega^2 x^3 = (1+x)(1+\omega x)^2$,

and $1 - \lambda x - \lambda\xi - \xi x = (1-x)(1-\omega x)^2$;

therefore $$(1-y^2) = \frac{(1-x^2)(1-\xi)^2}{(1-\lambda\xi)^2}.$$

Similarly, we have $1+\omega y^2 = \dfrac{(1-\lambda\omega^2x^2)^2+\omega x^2(\lambda+\omega^2x^2)^2}{(1-\lambda\xi)^2}$

$$= \frac{1+(\omega+2\omega^2)x^2+(\omega+2\omega^3)x^4+\omega^2x^6}{(1-\lambda\xi)^2} = \frac{(1+\omega x^2)(1+\xi)^2}{(1-\lambda\xi)^2};$$

hence $$\frac{dy}{\lambda dx} = \sqrt{\left(\frac{1-\xi}{1-\lambda\xi}\right)^2\left(\frac{1+\xi}{1-\lambda\xi}\right)^2} = \sqrt{\left(\frac{1-y^2}{1-x^2}\cdot\frac{1+\omega y^2}{1+\omega x^2}\right)}.$$

[If we put $1-\omega^2(\omega-\omega^2)x^2 = \mathrm{M}$, and observe that $(\omega-\omega^2)^2 = -3$, we obtain, by differentiating (1),

$$dy/dx = (\omega-\omega^2)(1-\omega x^4)/\mathrm{M}^2;$$

also, after simple reductions,

$$(1-y^2)^{\frac{1}{2}} = (1-x^2)^{\frac{1}{2}}(1-\omega^2x^2)/\mathrm{M}, \quad (1+\omega y^2)^{\frac{1}{2}} = (1+\omega x^2)^{\frac{1}{2}}(1+\omega^2x^2)/\mathrm{M},$$

whence from (1), by dividing, we obtain the required result.]

10593. (Professor CATALAN.)—Si, à un quadrilatère inscriptible Q, on inscrit une infinité de quadrilatères P dont le périmètre soit minimum, ce minimum commun est une quatrième proportionnelle au rayon et aux diagonales de Q.

Solution by Professors ANDERSON, BEYENS, *and others.*

Let ABCD, Fig. 1, be the quadrilateral inscribed in a circle. Take P any point on the side AD, and let P_1 be its image in AB, P_2 the image of P_1 in BC, and P_3 the image of P_2 in CD. Join PP_3 meeting CD in S,

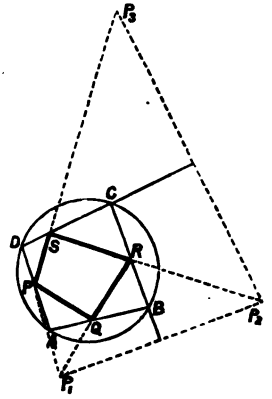


Fig. 1.

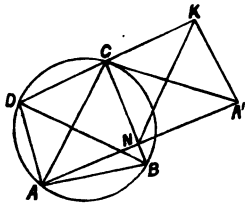


Fig. 2.

SP_2 meeting BC in R, RP_1 meeting AB in Q, and PQ. Then, as is

easily proved, PQRS is the inscribed quadrilateral of minimum perimeter which has an angular point at P. Also, $PP_3 = PS + SR + RQ + QP$ the perimeter.

If, now, any other point P′ be taken on AD, and the point P'_3 corresponding to P_3 be found, it is seen, without difficulty, that $P_3P'_3$ is equal and parallel to PP′. Hence PP_3 is the same for all points P on AD; and, since AD has a common point, A, with AB, it is the same also for points on AB, and likewise for BC and CD.

Take now, Fig. 2, the point A as the point. Draw AN perpendicular to BC, and take NA′ = AN. Also, draw A′K perpendicular to DC, and join NK. Then NK = ½ the perimeter of the inscribed quadrilateral.

But NK = CA′ sin C, since NCKA′ is inscribable in a circle. Therefore

$$NK = CA \sin C.$$

But, if r = radius of circle ABCD, $2r \sin C = BD$. Therefore $2r \,.\, NK = BD \,.\, AC$, which proves the theorem.

7099. (The Editor.)—Sum (1) the series

$$\frac{\sin^3\theta}{1^4} - \frac{\sin^3 3\theta}{3^4} + \frac{\sin^3 5\theta}{5^4} - \&c.$$

between the limits $\theta = 0$, $\theta = 2\pi$, and $\theta = \frac{1}{6}\pi$, $\theta = \frac{5}{6}\pi$; and hence show (2) that $\pi^4 = \frac{3 \,.\, 2^4}{11}\left(\frac{1}{1^4} - \frac{1}{3^4} + \frac{1}{5^4} - \frac{1}{7^4} + \&c.\right)$.

Generalised Solution by Professor Franz Rogel.

By means of Bernoulli's function (ϕ) invented by Raabe (*Crelle J.*, T. 42, p. 48), and accomplished by Schlömilch (*Compendium der höheren Analysis*, II., pp. 207–217), viz.,

$$\phi(\theta, 2n) = \theta^{2n} - n\theta^{2n-1} + \binom{2n}{2} B_1 \theta^{2n-2} - \binom{2n}{4} B_2 \theta^{2n-4} \ldots$$

$$\ldots + (-1)^n \binom{2n}{2n-2} B_{n-1} \theta^2$$

$$= (-1)^n B_n + (-1)^{n-1} 2 \frac{(2n)!}{\pi^{2n}} \sum_{\lambda=1}^{\infty} \frac{\cos 2\lambda\pi\theta}{(2\lambda)^{2n}}, \quad 0 \leq \theta \leq 1 \ldots\ldots(1),$$

$$\phi(\theta, 2n-1) = \theta^{2n-1} - n\theta^{2n-2} + \binom{2n}{2} B_1 \theta^{2n-3} - \binom{2n}{4} B_2 \theta^{2n-5} \ldots$$

$$\ldots + (-1)^n \binom{2n}{2n-2} B_{n-1} \theta$$

$$= (-1)^n 2 \frac{(2n-1)!}{\pi^{2n-1}} \sum_{\lambda=1}^{\infty} \frac{\sin 2\lambda\pi\theta}{(2\lambda)^{2n-1}}, \quad n > 1, \quad 0 \leq \theta \leq 1 \ldots\ldots(2),$$

each power of π can be expressed by an infinite series of powers with the variations of sign $+ + - -$.

Transforming these equations, in which B signifies BERNOULLI's numbers,

$$(-1)^{n-1}\left[\phi(\theta, 2n)+(-1)^{n-1}B_n\right]\frac{\pi^{2n}}{2(2n)!}=\sum_{\lambda=1}^{\infty}\frac{\cos 2\lambda\pi\theta}{(2\lambda)^{2n}}\quad\ldots\ldots(1'),$$

$$(-1)^{n}\phi(\theta, 2n-1)\frac{\pi^{2n-1}}{2(2n-1)!}=\sum_{\lambda=1}^{\infty}\frac{\sin 2\lambda\pi\theta}{(2\lambda)^{2n-1}}\quad\ldots\ldots(2');$$

hence we form

$$\left[(-1)^{n-1}\phi(\theta, 2n)+B_n\right]\frac{\pi^{2n}}{2(2n)!}-\frac{1}{2^{2n}}\left[(-1)^{n-1}\phi(2\theta, 2n)+B_n\right]\frac{\pi^{2n}}{2(2n)!}$$

$$=\frac{1}{2^{2n}}\sum_{\mu=1,3,5,7\ldots}^{\infty}\frac{\cos\mu\theta\,.\,2\pi}{\mu^{2n}}\quad\ldots\ldots(1''),$$

and

$$(-1)^{n}\frac{\pi^{2n-1}}{2(2n-1)!}\left[\phi(\theta, 2n-1)-\frac{1}{2^{2n-1}}\phi(2\theta, 2n-1)\right]$$

$$=\frac{1}{2^{2n-1}}\quad\frac{\sin\mu\theta\,.\,2\pi}{\mu^{2n-1}}\quad\ldots\ldots(2'').$$

If, in these equations, we put $\theta=\frac{1}{8}$, using the formulas

$$\cos\frac{\mu\pi}{4}=\pm\cos\frac{\pi}{4}=\pm\frac{1}{\sqrt{2}},\qquad\sin\frac{\mu\pi}{4}=\pm\sin\frac{\pi}{4}=\pm\frac{1}{\sqrt{2}},$$

we get

$$\pi^{2n}=\frac{(2n)!\sqrt{2}}{(-1)^{n-1}\left[2^{2n}\phi(\frac{1}{8}, 2n)-\phi(\frac{1}{4}, 2n)\right]+(2^{2n}-1)B_n}$$

$$\times\left\{\frac{1}{1^{2n}}-\frac{1}{3^{2n}}-\frac{1}{5^{2n}}+\frac{1}{7^{2n}}+\frac{1}{9^{2n}}--\ldots\right\}\quad\ldots\ldots(A),$$

$$\pi^{2n-1}=\frac{(2n-1)!\sqrt{2}}{2^{2n-1}\phi(\frac{1}{8}, 2n-1)-\phi(\frac{1}{4}, 2n-1)}$$

$$\times\left\{\frac{1}{1^{2n-1}}+\frac{1}{3^{2n-1}}-\frac{1}{5^{2n-1}}-\frac{1}{7^{2n-1}}++--\ldots\right\}\quad\ldots\ldots(B).$$

[Another solution of this Question 7099 is given on pp. 23, 24 of Vol. 39.]

10605. (The EDITOR.)—Through four given points draw (1) a conic such that one of its chords of intersection with a given conic may pass through a fifth given point, and show (2) what the problem becomes when two of the four points are the circular points at infinity.

Solution by Professor ANDERSON, M.A.

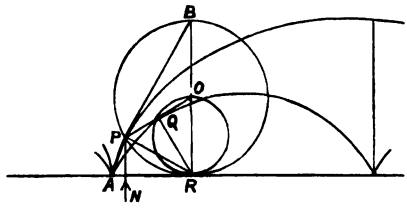
1. Let the four points be the points of intersection of the conics $U=0$, $V=0$. Any conic passing through the four points may be written $S\equiv U+\mu V=0$. Then, if the other conic be given by $W=0$, and if the chord of intersection of S and W pass through a fixed point x', y', z', we have $W'^3\Delta-W'^2S'\Theta+W'S'^2\Theta_1-S'^3\Delta_1=0$, where Δ, Δ_1 are the discriminants of S, W, and Θ, Θ_1 have their usual significations. It

is easily seen that the above equation is of the third degree in μ; and there are consequently three conics fulfilling the required conditions.

2. When two of the points are the circular points at infinity, the problem reduces to finding a circle which passes through two given points, and one of whose chords of intersection with a given conic passes through a third given point. It will be found, on forming the equation corresponding to that found for the general case, that there are three circles which satisfy the conditions.

10592. (Professor MORLEY.)—Rays of light parallel to an axis are reflected at a cycloid. Show that the caustic is a cycloid.

Solution by Professors ANDERSON, PUTNAM, *and others.*



Let BPR be the generating circle of the cycloid, and NP an incident ray parallel to the axis; then, since $\angle$ RPN = $\angle$ RPO,PO, the reflected ray, always passes through O, the centre of the circle. Draw a circle on OR as diameter meeting PO in Q, and join RQ. Then the arc RQ = arc RP = RA, and the point Q traces out a cycloid, the tangent to which is PO the reflected ray. The caustic is therefore a cycloid the diameter of whose generating circle is half that of the generating circle of the reflecting cycloid.

10623. (Professor LAMPE, LL.D.)—Solve, in positive integers,

$$(x-y)^4 = x^3+y^3.$$

Solution by R. W. D. CHRISTIE; Rev. J. L. KITCHIN; *and others.*

To solve $(x-y)^{n+1} = x^n+y^n$; we have $c = a^n+(a-1)^n$, where a is any integer whatever; hence $c^{n+1} = (ac)^n + \{(a-1)c\}^n$.

Let $x = ac$ and $y = (a-1)c$; thus, since a and c are known, the problem is solved.

10645. (J. O'BYRNE CROKE, M.A.) — The semi-traces of planes through the axes of x and y of an ellipsoid on the planes of yz, xz, respec-

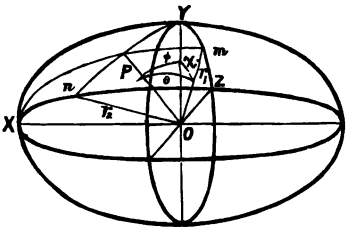
tively, are r_1 and r_2; if $f_1^2 = a^2 - r_1^2$, $f_2^2 = b^2 - r_2^2$, $F_1^2 = a^2 - c^2$, $F_2^2 = b^2 - c^2$; prove that the length of the radius vector, which is half the diameter in which they intersect, is given by

$$\rho^2 = \left\{a^2F_1^2/f_1^2 + b^2F_2^2/f_2^2 - (a^2+b^2-c^2)\right\} / \left\{F_1^2/f_1^2 + F_2^2/f_2^2 - 1\right\}.$$

Solution by the PROPOSER; REV. J. L. KITCHIN; *and others.*

The angle opposite ϕ in the spherical Δ (θ, ϕ, χ) being a right angle, we have

$$\cos\phi = \cos\chi\cos\theta.$$



Also, in the elliptic quadrant OXm we have

$$\rho^2\left(\frac{\sin^2\theta}{a^2} + \frac{\cos^2\theta}{a^2-f_1^2}\right) = 1;$$

and in OYm we have

$$\rho^2\left(\frac{\cos^2\phi}{b^2} + \frac{\sin^2\phi}{b^2-f_2^2}\right) = 1.$$

Again,
$$r_1^2\left(\frac{\cos^2\chi}{b^2} + \frac{\sin^2\chi}{c^2}\right) = 1;$$

therefore
$$\cos^2\chi = \left(\frac{1}{a^2-f_1^2} - \frac{1}{c^2}\right) \Big/ \left(\frac{1}{b^2} - \frac{1}{c^2}\right).$$

$$\rho^2\left\{\frac{1}{b^2-f_2^2} + \cos^2\phi\left(\frac{1}{b^2} - \frac{1}{b^2-f_2^2}\right)\right\} = 1;$$

therefore
$$\cos^2\phi = \left(\frac{1}{\rho^2} - \frac{1}{b^2-f_2^2}\right) \Big/ \left(\frac{1}{b^2} - \frac{1}{b^2-f_2^2}\right) = \&c.,$$

so
$$\cos^2\theta = \left(\frac{1}{\rho^2} - \frac{1}{a^2}\right) \Big/ \left(\frac{1}{a^2-f_1^2} - \frac{1}{a^2}\right),$$

therefore
$$\left(\frac{1}{\rho^2} - \frac{1}{b^2-f_2^2}\right) \Big/ \left(\frac{1}{b^2} - \frac{1}{b^2-f_2^2}\right)$$

$$= \left(\frac{1}{a^2-f_1^2} - \frac{1}{c^2}\right) \Big/ \left(\frac{1}{b^2} - \frac{1}{c^2}\right) \cdot \left(\frac{1}{\rho^2} - \frac{1}{a^2}\right) \Big/ \left(\frac{1}{a^2-f_1^2} - \frac{1}{a^2}\right).$$

Whence
$$\rho^2\left\{f_1^2(c^2-b^2) + f_1^2f_2^2 + f_2^2(c^2-a^2)\right\}$$
$$= a^2f_2^2(c^2-a^2) + b^2f_1^2(c^2-b^2) + f_1^2f_2^2(a^2+b^2-c^2);$$

therefore
$$\rho^2\left\{f_1^2F_2^2 + f_2^2F_1^2 - f_1^2f_2^2\right\}$$
$$= a^2f_2^2F_1^2 + b^2f_1^2F_2^2 - f_1^2f_2^2(a^2+b^2-c^2);$$

$$\therefore \quad \rho^2 = \left\{a^2F_1^2/f_1^2 + b^2F_2^2/f_2^2 - (a^2+b^2-c^2)\right\} / \left\{F_1^2/f_1^2 + F_2^2/f_2^2 - 1\right\}.$$

10602. (Professor COUPRAU.)—Si deux cordes AB, CD d'un cercle sont telles que les droites qui joignent A et B au milieu de CD soient symétriques par rapport à CD, les droites joignant C et D au milieu de AB sont également symétriques par rapport à AB.

Solution by G. E. CRAWFORD, B.A.; Prof. ANDERSON, M.A.; *and others.*

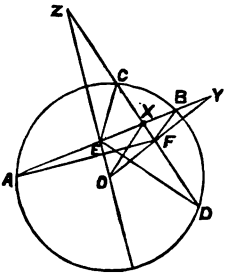
Draw perpendiculars from O on AB and CD meeting CD and AB in Z and Y; then C, E, D, Y are concyclic, since

$$\angle CEY = DEY,$$

and FY bisects CD at right angles; and Z, E, F, Y are concyclic. Also,

$$ZX \cdot XF = EX \cdot XY = CX \cdot XD$$
$$= AX \cdot XB;$$

therefore A, F, B, Z are concyclic, and, since EZ bisects AB at right angles, AF and BF are equally inclined to CD.

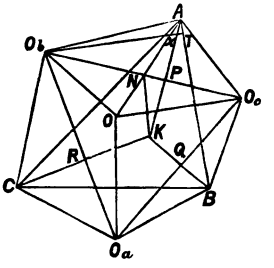


10622. (Professor NEUBERG.)—Soit K le point de Lemoine du triangle ABC, et soient O, O_a, O_b, O_c les centres des cercles circonscrits aux triangles ABC, KBC, KCA, KAB. Démontrer: (1) que O et K sont les foyers de l'ellipse qui touche les côtés du triangle $O_aO_bO_c$ en leurs milieux; (2) que les quadrilatères OBO_aC, OCO_bA, OAO_cB ont même surface.

Solution by Professors ANDERSON, ZERR, *and others.*

1. The points O_a, O_b, O_c are found by drawing lines perpendicular to KA, KB, KC (at their mid-points); and, clearly, OO_b, OO_a, OO_c are perpendicular to the sides of the triangle.

Draw O_bT perpendicular to AB meeting OA in X. Since O_bO is perpendicular to AC, O_bX to AB, and O_bP to AP, the angles OO_bN, NO_bX are equal, respectively, to CAK and KAB. But OA is perpendicular to an antiparallel drawn through P to the side BC. Hence, since this antiparallel is bisected at P, OX must be bisected at N; and therefore, since O_bX is parallel to OC_c, O_bO_c is also bisected at N.



Also, $ON + NK = OA$, and $\angle ONO_b = \angle KNP$. An ellipse whose foci are O and K, and major axis equal to OA, the circum-radius of the triangle, will consequently touch O_bO_c at its mid-point N. Similarly, it may be shown that the ellipse will also touch O_bO_a and O_aO at their mid-points.

2. Since $O_bN = O_cN$, the triangle O_bAO is equal to AO_cO. But these triangles are the halves, respectively, of the quadrilaterals O_bAOC and O_cAOB, which are therefore equal. In like manner it can be shown that the other quadrilateral is equal to either of these.

10647. (J. J. Barniville.)—P, Q are the mid-points of the sides BC, CD of a square ABCD; in AP, AM is taken = AB; AQ, MD meet in N: prove that $DN^2 = \frac{5}{9}MN^2$. In AC, AR is taken = AB; DR meets AQ in S: prove that $DS^2 = 2RS^2$. Prove also that, if U, V be the mid-points of AB, AD, then AP and AQ are each mutually trisected by UV and BD; and that consequently (1) UC, BD, PA, and (2) VC, BD, QA, are respectively concurrent.

Solution by R. Whapham, B.A.; H. J. Woodall; *and others.*

$$\alpha = \tan^{-1}\tfrac{1}{2} = \sin^{-1}\tfrac{1}{\sqrt{5}} = \cos^{-1}\tfrac{2}{\sqrt{5}},$$

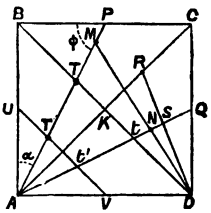
$$\phi = \tfrac{1}{2}\pi - \alpha;$$

$$\frac{DN}{AD} = \frac{\sin\alpha}{\sin AND}; \quad \frac{MN}{AM} = \frac{\sin(\phi-\alpha)}{\sin ANM};$$

$$\therefore \frac{DN}{MN} = \frac{\sin\alpha}{\sin(\phi-\alpha)} = \frac{\sqrt{5}}{3};$$

$$\therefore DN^2 = \tfrac{5}{9}MN^2,$$

$$\frac{DS}{RS} = \frac{\sin\alpha}{\sin(45°-\alpha)} = \sqrt{2}; \quad \therefore DS^2 = 2RS^2.$$



Again, $$TK = AK \tan(45°-\alpha) = \tfrac{1}{2}AK = \tfrac{1}{3}BK;$$

therefore $$BT = 2TK = Tt = Dt \text{ (by symmetry)};$$

therefore BD, and therefore UV, are trisected by AP and AQ. Now

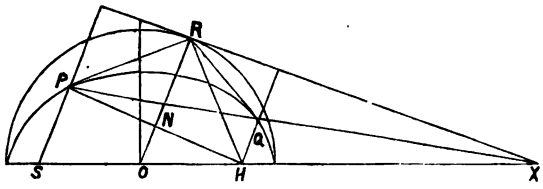
$$\frac{PT}{PB} = \frac{\sin 45°}{\sin(45°+\alpha)} = \frac{\sqrt{5}}{3}; \quad \therefore PT = \frac{\sqrt{5}}{3}PB = \frac{\sqrt{5}}{6}AB = \tfrac{1}{3}AP;$$

and from the triangles VAT', BPT, $AT' = PT = \frac{1}{3}AP$; therefore $TT' = \frac{1}{3}AP$; therefore AP, and therefore AQ, are trisected by BD, UV. Since U, K, P are the mid-points of the sides of the triangle ABC, therefore CU, BD, AP are concurrent. Similarly, CV, BD, AQ are concurrent.

10640. (Robert Bryant, D.Sc., B.A.)—The perpendiculars from the foci of an ellipse upon a tangent to the auxiliary circle cut the ellipse in two points on the same side of the axis major. Find geometrically the locus of the intersection of this chord with the aforenamed tangent.

Solution by Profs. ANDERSON, M.A.; CHAKRIVARTI, M.A.; *and others.*

Let the perpendiculars from the foci S, H of the ellipse on the tangent to the auxiliary circle at R meet the ellipse in the points P, Q, and let



PQ meet SH in X. PQ is clearly parallel to the diameter of the ellipse conjugate to OR. Join HR, RP.

We have, then, HN = NP; and

$NR = OR - ON = \frac{1}{2}(2OR - SP) = \frac{1}{2}HP = HN, \quad \therefore HN = NP = NR.$

Hence the angle HRP is right, and RP is therefore the tangent to the ellipse at P. In like manner, RQ is the tangent to the ellipse at Q, and therefore R is the pole of PQ.

Thus the polar of X with respect to the ellipse passes through R. But, since X lies on the production of the major axis, its polars with respect to the ellipse and circle coincide. Hence the polar of X with respect to the circle passes through R, and XR is, consequently, the tangent to the circle at R. Hence the locus of the intersection of PQ and the tangent to the auxiliary circle is the production of the major axis of the ellipse.

10711. (D. BIDDLE.)—Three weights m_1, m_2, m_3 are situated on three concentric circles, of radius r_1, r_2, r_3 respectively. Find the conditions of equilibrium about the common centre; and particularly give the angular relations when the weights are equal, and the radii as 3, 4, 5.

Solution by Professor SCHOUTE; J. D. H. DICKSON, M.A.; *and others.*

The angular relations are given by the equation

$$\frac{m_1 r_1}{\sin(r_2, r_3)} = \frac{m_2 r_2}{\sin(r_3, r_1)} = \frac{m_3 r_3}{\sin(r_1, r_2)}.$$

When the weights are equal, these equations show that the triangle with the sides r_1, r_2, r_3 indicates by its angles the supplements of the angles (r_2, r_3), (r_3, r_1), (r_1, r_2). In the case $r_1 = 3$, $r_2 = 4$, $r_3 = 5$, one of these angles is a right angle.

[If we draw the triangle $m_1 m_2 m_3$, the common centre of the circles is C.G. of weights, hence these weights are respectively proportional to the small triangles formed by joining each pair of weights to the centre of the circles. The particular solution for the radii 3, 4, 5 comes at once from the triangle whose sides are 3, 4, 5; viz., the angles are 90°, and the two angles greater than 90° and less than 180° given by $\sin^{-1}\frac{3}{5}$ and $\sin^{-1}\frac{4}{5}$.]

10709. (H. W. Segar, B.A.)—Solve the equation

$$(x^2-y^2-a^2)\,xdy+(y^2-x^2-a^2)\,ydx=0.$$

Solution by Professors Anderson, Chakrivarti, *and others.*

Let $u=y/x$, $w=xy$; then the equation becomes, by substitution,

$$(1/u^2-1)\,du-(a^2/w^2)\,dw=0,$$ from which $a^2-x^2-y^2=cxy$.

This is the equation of a system of conics whose centre is the centre of a square, and which passes through its angular points.

10710. (J. Brill, M.A.)—If A, B, C be the inverses of the points A′, B′, C′ with respect to their centroid O, prove that the product PA . PB . PC is stationary when P coincides with O.

Solution by Professor Morley, M.A.

Let $z_1'\,z_2',\,z_3'$ be the affixes of the points A′, B′, C′, and take O as origin.

Then $$\Sigma z_r'=0 \quad\ldots\ldots(1).$$

Inverting, let z_1, z_2, z_3 be the affixes of A, B, C; then

$$\Sigma 1/z_r=0 \quad\ldots\ldots(2).$$

It is to be noticed that, by writing $1/z$ for z, we get not the inverse group of a group of points, but the reflexion in the real axis of the inverse group. But there is rarely need to take this into account. From (2) (see Quest. 10495) O is a focus of the maximum inscribed ellipse of ABC. Which focus it is, one would like to know. Now let

$$w\equiv(z-z_1)(z-z_2)(z-z_3), \text{ so that } d/dt\,(\log w)=z\Sigma 1/(z-z_r)\ldots\ldots(3,4),$$

Hence, when $z=0$, w, and therefore its modulus $|w|$, is stationary.

And from (3) $$|w|=|z-z_1|\,.\,|z-z_2|\,.\,|z-z_3|=\text{PA}\,.\,\text{PB}\,.\,\text{PC}.$$

In fact, the curve PA . PB . PC = const., when it passes through O, passes twice through O in perpendicular directions. For, when z is small, the value of w is

$$w=w_0+1/2\,\frac{d^2w}{dz_0^2}\,z^2+\ldots,$$

since, from (4), $dw/dz_0=0$.

Hence, when w makes a half turn round w_0, z only makes a quarter turn round O. See Briot et Bouquet, "Théorie des Fonctions Elliptiques," or any elementary book on the function-theory. The question is of physical interest, on supposing equal infinitely long bars or vortex filaments placed vertically at ABC. (See Quest. 10562.) It seems an illustration of Earnshaw's Theorem (see Minchin's *Statics*, 2nd Ed., p. 428).

If instead of the centroid we start with a point whose areal coordinates referred to A′, B′, C′ are m_1, m_2, m_3, we shall obtain a similar result for $\text{PA}^{m_1}\,.\,\text{PB}^{m_2}\,.\,\text{PC}^{m_3}$, and the point got is a point of still water for vortices of strengths m_1, m_2, m_3, while the inverse point with regard to the triangle

of vortices is the other point of still water. And similarly for n points, if for inverse point we read focus of a curve of class $n-1$, with one focus at O, touching all the joins of the n points.

10656. (Professor EMMERICH.)—Prove that in any triangle, unless it be equilateral, $\cot\omega > \tan\frac{1}{2}A + \tan\frac{1}{2}B + \tan\frac{1}{2}C$.

Solution by Professors ANDERSON, ZERR, *and others.*

If we substitute for $\cot\omega$ and for the tangents of the half-angles their values in terms of the sides, the inequality becomes

$$a(a-b) + b(b-c) + c(c-a) \geqq 0.$$

Let $x = b+c-a, \quad y = a+c-b, \quad z = a+b-c;$

then x, y, z are positive quantities, and by substitution the inequality becomes

$$x^2 + y^2 + z^2 \geqq xy + yz + xz,$$

which is true, since

$$x^2 + y^2 \geqq 2xy, \quad x^2 + z^2 \geqq 2xz, \quad y^2 + z^2 \geqq 2yz.$$

10697. (Professor DE LONGCHAMPS.)—Soit BB′ le petit axe d'une ellipse Γ; du point B, abstraction faite de BB′, on peut mener à Γ deux normales. Soit C le pied d'une de ces normales. En désignant par F l'un des foyers de Γ, la perpendiculaire élevée en F à BF rencontre la tangente, en B, en un certain point D. Démontrer que BD = BC.

Solution by Professors SCHOUTE, ANDERSON, *and* G. G. MORRICE, M.B.

Dans le triangle rectangulaire BFD on a BF = a, tandis que la projection de BF sur BD est $c = (a^2 - b^2)^{\frac{1}{2}}$. Donc BD = $BF^2/BE = a^2/c$. Et l'élimination de x entre les équations

$$x^2/a^2 + y^2/b^2 = 1, \quad x^2 + (y \pm b)^2 = a^4/c^2$$

de l'ellipse et du cercle à centre B et à rayon a^2/c fait trouver deux parallèles à l'axe des x, qui coincident—c'est à dire, $y = \pm b^3/c^2$, &c.

10576. (D. EDWARDES, B.A.)—The boundaries of a membrane are two confocal ellipses, and are fixed in space. The density at any point varies inversely as the product of the distances of the point from the foci.

Show that there are modes of vibration in which the nodal lines are ellipses confocal with the boundaries, the periods being given by

$$P = \frac{1}{i\pi}\left(\frac{2\pi M}{T}\log\frac{a+b}{a'+b'}\right)^{\frac{1}{2}},$$

where i is an integer, M the mass of the membrane, T the tension, a, b and a', b' are the semi-axes of the outer and inner boundaries.

Solution by Professor ZERR; *and others.*

Taking elliptics (ξ, η), where $x = c\cos h\eta\cos\xi$, $y = c\sin h\eta\sin\xi$, we have $\rho c^2(\cos^2 h\eta - \cos^2\xi) = \kappa$, $M = \iint\rho\,dx\,dy = \kappa\iint d\xi\,d\eta$, the limits of η being η_1, η_2; of ξ, $2\pi, 0$. Then

$$M = 2\pi\kappa(\eta_1 - \eta_2) \text{ and } \rho = M/\{2\pi(\eta_1-\eta_2)c^2(\cos^2 h\eta - \cos^2\xi)\}.$$

The equation of motion being $\frac{d^2w}{dt^2} = \frac{T}{\rho}\left(\frac{d^2w}{\theta x^2} + \frac{d^2w}{\theta y^2}\right)$, we have in virtue of the equations $\frac{dy}{d\xi} = -\frac{dx}{d\eta}$, $\frac{dx}{d\eta} = \frac{dy}{d\xi}$, that

$$\frac{d^2w}{dx^2} + \frac{d^2w}{dy^2} = \left(\frac{d^2w}{d\xi^2} + \frac{d^2w}{d\eta^2}\right) \Big/ c^2(\cos^2 h\eta - \cos^2\xi);$$

and if the nodal lines are confocal ellipses, taking ξ constant, the equation becomes $\frac{d^2w}{dt^2} = \alpha^2\frac{d^2w}{d\eta^2}$, where $\alpha^2 = 2\pi T(\eta_1-\eta_2)/M$, the solution of which, under the given conditions, is $w = \sin i\pi\,.\,\frac{\eta-\eta_2}{\eta_1-\eta_2}\sin\frac{i\pi\alpha t}{\eta_1-\eta_2}$, and the periodic time is $\frac{2(\eta_1-\eta_2)}{i\alpha}$, which reduces to the form in the question, remarking that $e^{\eta_1} + e^{-\eta_1} = 2a/c$, $e^{\eta_1} - e^{-\eta_1} = 2h/c$, &c.

10591. (Professor ZERR.)—Prove that the area of the triplicate-ratio circle is equal to the areas of the cosine and the Brocard circles.

Solution by J. J. BARNIVILLE; Professor ANDERSON; *and others.*

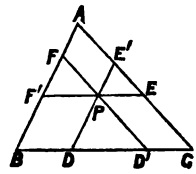
Let P be the symmedian point, and DE′, EF′, FD′ parallels to the sides. Then

$$D'E^2 = DD'.EE'.FF' = a^3b^3c^3/(\Sigma a^2)^3.$$

But the triangles PDF′ and PED are each equiangular to ABC; hence

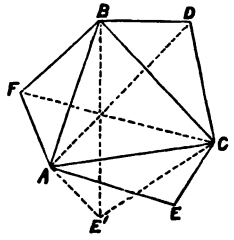
$$F'P.PE = D'E^2,$$

i.e., $$\rho_a^2 - \rho_b^2 = \frac{a^2b^2c^2}{(a^2+b^2+c^2)} = \rho_c^2.$$



10080. (Professor Syamadas Mukhopâdhyây, B.A. Extension of Quest. 9849.)—Similar triangles are similarly drawn on the sides of a polygon as bases, all externally or all internally. If equal particles be placed at their vertices, the C.G. of the system will coincide with the C.G. of equal particles placed at the angular points of the polygon.

Solution by Syadamas *and* Asutosh Mukhopâdhyây.



It will be sufficient to prove this in the case of a triangle, for it is easily seen that, if this proposition holds for a polygon of n sides, it will also hold for a polygon of $n+1$ sides.

ABC is a triangle, D, E, F are vertices of triangles drawn on BC, CA, AB. Invert the triangle ACE, and replace it on its base as ACE′. Join BE′, CF, AD.; then

$\angle BCE' = DCA$, $BC \cdot CE' = DC \cdot CA$,

therefore $\triangle BCE' = DCA$;

similarly, $\triangle BAE' = FAC$,

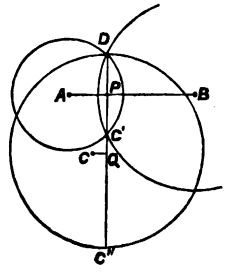
so that $\triangle DCA + FAC = BCE' + BAE' = ABC + AEC$.

But these triangles have a common base AC, therefore algebraic sum of perpendiculars on AC from A, B, C and D, E, F are equal. The same equality holds for perpendiculars on any other side BC.

Therefore the C.G. of equal particles at A, B, C must coincide with the C.G. of equal particles on D, E, F.

10699. (Professor Neuberg.)—Soient A, B, C les centres de trois circonférences qui passent par un même point D et se coupent, en outres deux à deux, aux points A′, B′, C′. Les cordes DA^1, DB^1, DC^1 communes à deux des circonférences rencontrent la troisième circonférence aux points A″, B″, C″. Démontrer que les droites A′A″, B′B″, C′C″ sont doubles des hauteurs du triangle ABC.

Solution by Professors Schoute, Aiyar, *and others.*



Comme AB est perpendiculaire à DC′, le segment C′C″ a la direction de l'hauteur du triangle ABC menée par le sommet C. Si P est le point d'intersection de AB et DC′, et Q la projection de C sur DC′, on a

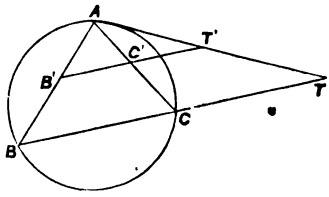
$$2DP = DC', \quad DQ = DC''.$$

Donc, par soustraction,

$$2PQ = C'C'', \text{ \&c.}$$

10355. (Colonel THIRY.)—Dans un triangle ABC, si AT est la tangente au cercle circonscrit, et si une sécante quelconque coupe AT, AB, AC en des points T′, B′, C′, démontrer que AB : AC′−AC : AB′ = BC : AT′.

Solution by R. KNOWLES, B.A.; Prof. IGNACIO BEYENS; *and others.*



The trilinear equations to AT and the secant are

$cy + bz = 0$, $lx + my + nz = 0$;

the coordinates of B′, C′, T′ are readily found, and thence the lengths of AB′, AC′, AT′,

$bcl/(bl - am)$, $bcl/(cl - an)$,

$abcl/(c^2l - b^2l + abm - acn)$,

$\therefore$ AB : AC′−AC : AB′ = $(c^2l - b^2l + abm - acn)/(bcl)$ = BC : A′T′.

10351. (Professor SYAMADAS MUKHOPÂDHYÂY, B.A.)—An infinite number of lines are drawn in a plane at random except in so far that they make with a fixed straight line an angle which only varies from α to $\alpha + \psi$, ψ being less than π. Show (1) that they intersect within any closed figure of area S at $N^2(\psi - \sin\psi)S$ points, where N is the number of these lines intersecting a convex figure of unit perimeter when they are not restricted in direction, that is, when $\psi = \pi$; and hence (2) deduce a simple proof of Professor CROFTON's celebrated theorem.

Solution by the PROPOSER.

Let P be the number of unrestricted random lines which intersect a unit length normally, and Q the number of variety of directions for these lines per unit angle.

Then, if a line of length p make with the fixed straight line an angle θ intermediate between α and $\alpha + \psi$, the number of restricted random lines which intersect p is

$$\int_{\alpha}^{\theta} Pp \sin(\theta - \phi) \cdot \theta d\phi + \int_{\theta}^{\alpha+\psi} Pp \sin(\phi - \theta) \cdot Q d\phi$$

$$= PQp\{2 - \cos(\theta - \alpha) - \cos(\alpha + \psi - \theta)\} = P.Q.p.x, \text{ suppose.}$$

If $\psi = \pi$, this number becomes PQ . $2p$. Then, because a line can intersect a convex figure at only two points, the number of unrestricted random lines which intersect a convex polygon is $\frac{1}{2}$ PQ . 2 (sum of the sides); hence any convex figure of perimeter l is intersected by PQ . l or Nl random lines.

Let this p be a chord of the closed figure whose area is S. It will be intersected by the restricted random lines at Npx points. Varying p

across the figure without varying θ, we get PS . Nx points of intersection; and now varying θ from α to $\alpha+\psi$, we get

$$\int_{\alpha}^{\alpha+\psi} \mathrm{P}\,.\,\mathrm{S}\,.\,\mathrm{N}x\,.\,Qd\theta = \mathrm{N}^2\mathrm{S}\int_{\alpha}^{\alpha+\psi} xd\theta = \mathrm{N}^2\mathrm{S}\left\{2\psi - 2\sin\psi\right\}.$$

Now evidently we have counted each point of intersection twice, hence the total number of points of intersection within the closed figure of area S is $\mathrm{N}^2(\psi-\sin\psi)\,\mathrm{S}$.

If $\psi=\pi$, this number becomes $\mathrm{N}^2\mathrm{S}\pi$. The total number of unrestricted random lines which intersect a convex figure of perimeter l is Nl, as has been proved. These intersect with each other at $\frac{1}{2}\mathrm{N}l(\mathrm{N}l-1)$ or $\frac{1}{2}\mathrm{N}^2l^2$ points (since N is negligable compared with N^2). Therefore these intersect outside the convex figure at $\frac{1}{2}\mathrm{N}^2l^2-\mathrm{N}^2\mathrm{S}\pi$ points.

Hence $$\int \mathrm{N}^2(\psi-\sin\psi)\,d\sigma = \tfrac{1}{2}\mathrm{N}^2l^2-\mathrm{N}^2\mathrm{S}\pi,$$

where σ is an element of outside area, and ψ the angle which the convex figure subtended at σ. Removing N^2, we get Crofton's Theorem.

10733. (R. Tucker, M.A.)—A′B′C′, A″B″C″, are circum-triangles of ABC, so drawn that A′B′, A″C″ pass through A and are perpendicular respectively to ΩA, Ω′A, &c.; prove that, if Ω_1 (A′B′C′), Ω_1', (A″B″C″) are the other Brocard-points of the circum-triangles, then the four Brocard-points form a rectangle of which the circum-circle is (5) of Quest. 10139 (see *Reprint*, Vol. LI., p. 119) and the Brocard-points (above) are the Brocard-points of the diametral triangle of ABC. [For general properties of the above triangles, see Simmons's *Recent Geometry*, p. 178.]

Solution by the Proposer.

Since A, Ω, B, B′ lie on a circle, therefore $\angle \mathrm{BB'}\Omega = \angle \mathrm{BA}\Omega$; hence Ω is positive B.-point of A′B′C′.

Similarly, Ω′ is negative B.-point of A″B″C″.

Now $\angle \mathrm{AB'}\Omega_1 = \omega = \angle \Omega\mathrm{AB}$, therefore $\mathrm{B'}\Omega_1$ is perpendicular to AB and $\mathrm{AB'} = c\cos(\mathrm{B}-\omega)/\sin\mathrm{B} = \mathrm{A}\Omega\cot(\mathrm{B}-\omega)$; hence projection of AB′ on $\mathrm{AB} = \mathrm{AB'}\sin\omega = \mathrm{B}\Omega\cos(\mathrm{B}-\omega)$ = projection of ΩB on AB, and, therefore, Ω_1 (and similarly Ω_1') are easily found.

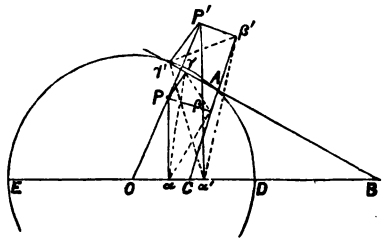
Again, $\mathrm{AA'} = \mathrm{A}\Omega\cot\omega$, therefore $\mathrm{A'B'} = \mathrm{A}\Omega\sin\mathrm{B}/\sin\omega\sin(\mathrm{B}-\omega) = c\operatorname{cosec}\omega$; hence modulus of similarity for A′B′C′ (and A″B″C″) is $\operatorname{cosec}\omega$. From above the projections of mid-point of $\Omega\Omega_1$ (and of $\Omega'\Omega_1'$) upon the sides bisect the sides of ABC; therefore the mid-point is the circum-centre of ABC, hence the results stated immediately follow.

The trilinear coordinates of Ω_1, Ω_1' are seen to be, respectively, $(b\cos\mathrm{A}-c\sin^2\omega)/\sin\mathrm{B}$, ..., ..., and $(c\cos\mathrm{A}-b\sin^2\omega)/\sin\mathrm{C}$, ..., ...

From the above and Quest. 10139 we see that Ω, Ω′, Ω_1, Ω_1' are B.-points (in pairs) of six triangles, viz., of the above three and of the three got by forming the diametral triangle and its corresponding associated triangles.

10695. (Professor Neuberg.)—Soient P, P′ deux points inverses par rapport à un cercle d'Apollonius du triangle ABC. Démontrer que les triangles podaires de ces points par rapport au triangle ABC sont inversement semblables.

Solution by Professor Schoute.



Le cercle d'Apollonius du triangle ABC par rapport à BC passe par A et a son centre O au milieu du segment DE déterminé sur BC par les bissectrices de l'angle A. On a donc, P et P′ étant deux points inverses par rapport à ce cercle, les relations

$$OA^2 = OB \cdot OC = OP \cdot OP',$$

ce qui mène à une foule de couples de triangles semblables. Ces triangles donnent

$$\frac{AP}{AP'} = \frac{OP}{OA} = \frac{OA}{OP'}, \quad \frac{AC}{AB} = \frac{OA}{OB}, \quad \frac{BP}{CP'} = \frac{OP}{OC}, \quad \frac{CP}{BP'} = \frac{OC}{OP'}.$$

De plus les quadrilatères inscriptibles $P\beta A\gamma$, &c., donnent

$$\left.\begin{aligned} \beta\gamma &= AP \sin A \\ \gamma\alpha &= BP \sin B \\ \alpha\beta &= CP \sin C \end{aligned}\right\}, \quad \left.\begin{aligned} \beta'\gamma' &= AP' \sin A \\ \gamma'\alpha' &= BP' \sin B \\ \alpha'\beta' &= CP' \sin C \end{aligned}\right\},$$

donc

$$\frac{\beta\gamma}{\beta'\gamma'} = \frac{AP}{AP'},$$

$$\frac{\gamma\alpha}{\alpha'\beta'} = \frac{BP \sin B}{CP' \sin C} = \frac{BP}{CP'} \cdot \frac{AC}{AB} = \frac{OP}{OC} \cdot \frac{OA}{OB} = \frac{OP}{OA},$$

$$\frac{\alpha\beta}{\gamma'\alpha'} = \frac{CP \sin C}{BP' \sin B} = \frac{CP}{BP'} \cdot \frac{AB}{AC} = \frac{OC}{OP'} \cdot \frac{OB}{OA} = \frac{OA}{OP'}.$$

Conséquemment

$$\frac{\beta\gamma}{\beta'\gamma'} = \frac{\gamma\alpha}{\alpha'\beta'} = \frac{\alpha\beta}{\gamma'\alpha'}.$$

La Question 10695 forme une partie de la proposition suivante:—

"Il y a 12 points, dont les triangles podaires par rapport à ABC sont semblables à un triangle donné $\alpha\beta\gamma$ (*Verslagen en mededeelingen*, Amsterdam, 1886, t. III$_3$, p. 50). Je les indique par le tableau

$$\begin{aligned} &P_1(\alpha_1\beta_1\gamma_1), \quad P_2(\beta_2\gamma_2\alpha_2), \quad P_3(\gamma_3\alpha_3\beta_3) \ldots (++), \\ &P_4(\alpha_4\gamma_4\beta_4), \quad P_5(\beta_5\alpha_5\gamma_5), \quad P_6(\gamma_6\beta_6\alpha_6) \ldots (--), \\ &P_1'(\alpha_1'\beta_1'\gamma_1'), \quad P_2'(\beta_2'\gamma_2'\alpha_2'), \quad P_3'\ \gamma_3'\alpha_3'\beta_3') \ldots (+-), \\ &P_4'(\alpha_4'\gamma_4'\beta_4'), \quad P_5'(\beta_5'\alpha_5'\gamma_5'), \quad P_6'(\gamma_6'\beta_6'\alpha_6') \ldots (-+). \end{aligned}$$

Dans ce tableau les sommets homologues à α, β, γ occupent respectivement le premier, le second, le troisième rang; α_i et α_i' sont sur BC, β_i

et β'_i sur CA, γ_i et γ'_i sur AB. Chaque point P a deux signes, dont le premier se rapporte à la permutation des lettres $\alpha_i\beta_i\gamma_i$, tandis que le second indique si $\alpha_i\beta_i\gamma_i$ est de même orientation que ABC ou d'orientation différente.

Cela posé, on trouve que P_i et P'_i sont des points inverses par rapport au cercle circonscrit (*l.c.*). De plus les couples de points des tableaux suivants:—

(A)	(B)	(C)
$P_1, P_4; \quad P'_1, P'_4$	$P_1, P_6; \quad P'_1, P'_6$	$P_1, P_5; \quad P'_1, P'_5$
$P_2, P_6; \quad P'_2, P'_6$	$P_2, P_5; \quad P'_2, P'_5$	$P_2, P_4; \quad P'_2, P'_4$
$P_3, P_5; \quad P'_3, P'_5$	$P_3, P_4; \quad P'_3, P'_4$	$P_3, P_6; \quad P'_3, P'_6$

sont des points inverses par rapport au cercle D'APOLLONIUS passant par le point placé a la tête.

On peut donc passer d'un quelconque des 12 points à chacun des 11 autres à l'aide de l'inversion par rapport aux quatre cercles indiqués."

Les points $P_1, P_2, P_3, P_4, P_5, P_6$ sont sur un même cercle (*l.c.*); de même les points P accentués.

10719. (Professor HUDSON, M.A.)—Supposing that the prosperity of a country varies as the excess of the average individual wealth over a given sum, and that by defeat and civil war one-mth of the wealth and one-nth of the population are destroyed, and the nation has to pay a property tax of $100r$ per cent. to the victor: show that the ruling Government must now put to death about 8 per cent. of the original population in order that the prosperity may be the same as before.

Solution by D. BIDDLE; Rev. J. L. KITCHIN; *and others.*

The "given sum" of the question may be left out of account. Let M = the total wealth, and N the total population prior to the war. Then we have the following equation, in which x is the required number to be slain:

$$(1-r)\,\mathrm{M}\left(1-\frac{1}{m}\right) \Big/ \left\{\mathrm{N}\left(1-\frac{1}{n}\right)-x\right\} = \frac{\mathrm{M}}{\mathrm{N}},$$

whence

$$\frac{x}{\mathrm{N}} = r\left(1-\frac{1}{m}\right)+\frac{1}{m}-\frac{1}{n}.$$

Thus, if $\quad \frac{1}{m}=\frac{1}{30}, \ \frac{1}{n}=\frac{1}{1000}$, and $r=\frac{1}{20}$ (so that $100r=5$),

we have

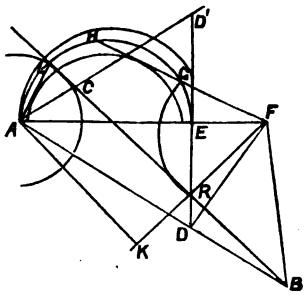
$$\frac{x}{\mathrm{N}} = \left\{\frac{29}{600}+\frac{97}{3000}\right\} = \cdot 0806,$$

whence the result follows.

10677. (The Rev. T. W. Openshaw.)—Given the vertical angle of a tirangle, its altitude, and the sum or difference of the sides; construct the triangle.

Solution by the Rev. J. R. Lunn, B.D.; D. Biddle; *and others.*

Bisect the vertical angle CAB by CF; let AD = ½ sum of sides, draw DF perpendicular to AB; then F is on the circumcircle of required triangle. Draw DE perpendicular to AF, on AE as diameter describe circle, and in it place AL = required altitude; draw a circle concentric with former, and FGH a tangent to this; then HG = altitude AL. With centre F, and FG, FH as radii describe circles cutting ED in R, and FR in K; through R draw BC perpendicular to FR; then ABC is required triangle.



For $$FG.FH = FR.FK = FE.FA,$$

and angle FER is right, therefore FKA is right, therefore

$$RK = HG = AL = \text{altitude};$$

also FRDB is cyclic, therefore

$$FBR = FDR = FAD = FAC,$$

therefore FBAC is cyclic; therefore ABC is triangle required. The proof for the given difference of sides is similar.

[Let CB be the common tangent to the parabola with focus F, vertex E, touching AC, AB, and the circle with centre A, and radius equal to the given altitude; then, since DB = CD', AD = ½ sum of sides; hence we have a construction for the common tangent to a parabola, and a circle with its centre on the axis of the parabola.]

10642. (W. J. C. Sharp, M.A.)—If V be the volume of the tetrahedron of reference; Δ_1, Δ_2, Δ_3, Δ_4 the areas of its faces; A', B', C', D' the projections of a point P, the tetrahedral content coordinates of which are λ', μ', ν', π', upon the faces; show that (1) the content of the tetrahedron PB'C'D' is $\frac{3^4}{2^2} \cdot \frac{\mu'\nu'\pi'}{\Delta_1^2 . \Delta_2^2 . \Delta_3^2} V^5$; and hence (2) the points A', B', C', D' all lie in the same plane, if

$$\frac{\Delta_1^2}{\lambda'} + \frac{\Delta_2^2}{\mu'} + \frac{\Delta_3^2}{\nu'} + \frac{\Delta_4^2}{\pi'} = 0.$$

Solution by Professors ANDERSON, SWAMINATHA AIYAR, *and others.*

1. The volume of the tetrahedron PB′C′D′ is, by a well-known theorem,

$$\tfrac{1}{6}PB'.PC'.PD'\left\{1-\cos^2(BC)-\cos^2(CD)-\cos^2(BD)\right.$$
$$\left.-2\cos(BC)\cos(CD)\cos(BD)\right\}^{\frac{1}{2}},$$

where (BC) signifies the supplement of the angle between PB′ and PC, or the angle between the faces ACD, ABD. We have, also,

$$PB'.\Delta_2 = 3\mu'V, \quad PC'.\Delta_3 = 3\nu'V, \quad PD'.\Delta_4 = 3\pi'V,$$

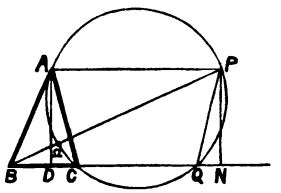
$\left\{1-\cos^2(BC)-\cos^2(CD)-\cos^2(BD)-2\cos(BC)\cos(CD)\cos(BD)\right\}^{\frac{1}{2}}$ is = 2N, where N is the second Staudtian of the solid angle A. Hence volume of $PB'C'D' = \dfrac{3^2.\mu'\nu'\pi'}{\Delta_2\Delta_3\Delta_4}V^3.N$. But $N = 9V^2/4\Delta_2\Delta_3\Delta_4$. Therefore volume of $PB'C'D' = \dfrac{3^4}{2^2}.\dfrac{\mu'\nu'\pi'}{\Delta_2^2\Delta_3^2\Delta_4^2}V^5$.

2. If we equate the sum of the volumes of the tetrahedron PB′CD′, PA′C′D′, PA′B′D′, PA′B′C′ to zero, we get the condition given.

10737. (R. F. DAVIS, M A.)—Give a *geometrical* proof of the formula $\cot\omega = \cot A + \cot B + \cot C$.

Solution by the PROPOSER.

Let ABC be a triangle; and let a circle be described touching AB in A, and passing through C. Then, if AP be the chord of the circle parallel to BC, BP intersects the circle in Ω, the positive Brocard-point, and the angle PBC = ω (Vol. XLVII., App. II., p. 9).



Let BC meet the circle again in Q; draw AD, PN perpendiculars upon BC. Since AP, CQ are parallel chords,

$$AD = PN, \quad DC = QN; \quad \text{therefore } BD + DC + DQ = BN.$$

Dividing each side by AD or PN, $\cot B + \cot C + \cot A = \cot\omega$, for angle CAB = AQC in the alternate segment.

10747. (Rev. T. C. SIMMONS, M.A.)—A triangle is formed by three straight lines chosen at random. Show that the chance of its being acute-angled is $\frac{1}{4}(4-\pi) = .4292$ nearly.

Solution by D. BIDDLE.

Let the longest side = unity, let x = the side next in length, and y = the shortest side. Then, for *any* triangle, the range of x is between

$\frac{1}{2}$ and 1, and of y, between $1-x$ and x. But for acute-angled triangles, the ranges of x, y, are $(\sqrt{\frac{1}{2}}, 1)$, $[(1-x^2)^{\frac{1}{2}}, x]$; hence

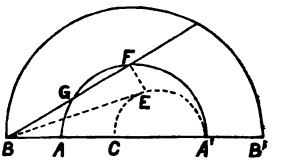
$$P = \left(\int_{\sqrt{\frac{1}{2}}}^{1}\int_{(1-x^2)^{\frac{1}{2}}}^{x} dx\,dy\right) \Big/ \left(\int_{\frac{1}{2}}^{1}\int_{1-x}^{x} dx\,dy\right)$$

$$= \left\{\int_{\sqrt{\frac{1}{2}}}^{1} [x-(1-x^2)^{\frac{1}{2}}]\,dx\right\} \Big/ \left\{\int_{\frac{1}{2}}^{1} (2x-1)\,dx\right\}$$

$$= \left\{(\tfrac{1}{2}-\tfrac{1}{4})-(\tfrac{1}{8}\pi-\tfrac{1}{4})\right\} / \left\{(1-\tfrac{1}{4})-(1-\tfrac{1}{2})\right\} = \tfrac{1}{2}(4-\pi).$$

9891. (R. W. D. Christie.)—Draw a straight line cutting two concentric circles, so that the part intercepted by them is divided into three equal portions.

Solution by J. C. St. Clair.



Let BAA'B' be any common diameter of the given circles. Take on it AC = BA, and upon A'C describe a semicircle. Let BE be the tangent to this circle from B, and on the circle (A) take F, so that BF = BE. Then BF is the required line. For let BF meet the circle (A) again in G; then

$(BF)^2 = (BE)^2 = BC \cdot BA' = 2BA \cdot BA'$,

and $BF \cdot BG = BA \cdot BA'$, $\therefore (BF)^2 = 2BF \cdot BG$ and $BF = 2BG$.

Hence the line is divided into three equal segments.

10653. (Professor Catalan.)—Décomposer, en une somme de deux carrés, le nombre $N = 17^3 \cdot 73^5 = 10\,185\,000\,736\,409$. [See Quest. 9967.]

Solution by R. W. D. Christie; Professor Zerr; *and others.*

We have

$$x^2+y^2 = (a^2+b^2)^3 = \left\{a(a^2-3b^2)\right\}^2 + \left\{b(3a^2-b^2)\right\}^2$$
$$= \left\{a(a^2+b^2)\right\}^2 + \left\{b(a^2+b^2)\right\}^2;$$

$$X^2+Y^2 = (a^2+b^2)^5 = \left\{a(a^4-10a^2b^2+5b^4)\right\}^2 + \left\{b(5a^4-10a^2b^2+b^4)\right\}^2$$
$$= \left\{a(a^2+b^2)^2\right\}^2 + \left\{b(a^2+b^2)^2\right\}^2$$
$$= \left\{a(a^2+b^2)(3b^2-a^2)\right\}^2 + \left\{b(a^2+b^2)(3a^2-b^2)\right\}^2.$$

Here let $a = 4$, $b = 1$, and $a = 8$, $b = 3$, respectively, and we have at once these 12 resolutions, which are all:—

$$\begin{aligned}
17^3 \cdot 73^5 &= 3\,092\,572^2 + 788\,035^2 = 2\,357\,900^2 + 2\,150\,653^2 \\
&= 362\,372^2 + 3\,170\,755^2 = 1\,811\,860^2 + 2\,627\,197^2 \\
&= 3\,190\,628^2 + 69\,955^2 = 2\,848\,180^2 + 1\,439\,747^2 \\
&= 3\,099\,580^2 + 560\,003^2 = 1\,068\,428^2 + 3\,007\,235^2 \\
&= 2\,835\,028^2 + 1\,465\,475^2 = 1\,172\,380^2 + 2\,968\,253^2 \\
&= 2\,782\,340^2 + 1\,565\,197^2 = 1\,835\,572^2 + 2\,612\,685^2.
\end{aligned}$$

10618. (H. Brocard.)—Les rayons vecteurs MF, MF′ d'un point d'une ellipse rencontrent la courbe en P, P′; trouver le lieu (1) du pôle de PP′, et du point de rencontre (2) de PP′ avec la tangente menée en M, (3) des perpendiculaires élevées en F sur FM, et en F′ sur F′M.

Solution by Professor Anderson; Rev. J. L. Kitchin; *and others.*

1. Let the eccentric angles of the points M, P, P′ be α, β, γ. Since the chord MP passes through F, whose coordinates are $-c$, 0, we have the equation $\tan\frac{1}{2}\alpha \tan\frac{1}{2}\beta = -\frac{a+c}{a-c}$; and, since MP′ passes through F′, $\tan\frac{1}{2}\alpha \tan\frac{1}{2}\gamma = -\frac{a-c}{a+c}$. The coordinates of the pole of PP′, whose equation is $x\cos\frac{1}{2}(\beta+\gamma)/a + y\sin\frac{1}{2}(\beta+\gamma)/b = \cos\frac{1}{2}(\beta+\gamma)$,

are $$x = a\frac{1-\tan\frac{1}{2}\beta\tan\frac{1}{2}\gamma}{1+\tan\frac{1}{2}\beta\tan\frac{1}{2}\gamma},\quad y = b\frac{\tan\frac{1}{2}\beta+\tan\frac{1}{2}\gamma}{1+\tan\frac{1}{2}\beta\tan\frac{1}{2}\gamma},$$

or, denoting $\tan\frac{1}{2}\alpha$ by $1/\mu$,

$$x = a(1-\mu^2)/(1+\mu^2),\ y = -\frac{2\mu}{1+\mu^2}\cdot\frac{2a^2-b^2}{b}.$$

Eliminating μ, the locus is found to be $y^2b^2/(2a^2-b^2)^2 + x^2/a^2 = 1$.

2. The equation of the tangent at M is $x\cos\alpha/a + y\sin\alpha/b = 1$, or $x/a(\mu^2-1) + (2\mu/b)y = \mu^2+1$, and that of the chord PP′ is

$$x(1-\tan\tfrac{1}{2}\beta\tan\tfrac{1}{2}\gamma)/a + y(\tan\tfrac{1}{2}\beta+\tan\tfrac{1}{2}\gamma)/b = 1+\tan\tfrac{1}{2}\beta\tan\tfrac{1}{2}\gamma;$$

or,
$$(1-\mu^2)\,x/a - \frac{2\mu y}{b}\cdot\frac{a^2+c^2}{a^2-c^2} = 1+\mu^2.$$

Eliminating μ, as before, we get $(a^3/x)^2 + (b^3/y)^2 = (a^2-b^2)^2$.

3. Coordinates of this point in terms of the coordinates x', y' of the point M are $x = -x'$, $y = -(c^2-x'^2)/y'$. Its locus is easily found to be given by the equation $b^2y^2(x^2-a^2) + a^2(a^2-b^2-x^2)^2 = 0$.

10494. (Prof. WOLSTENHOLME, Sc.D.)—Prove that (1) the Lemoine-point of a maximum triangle inscribed in the ellipse $a^2y^2+b^2x^2=a^2b^2$ is given by any two of the three equations

$$\frac{x}{a}\cos\alpha+\frac{y}{b}\sin\alpha+\frac{a^2-b^2}{2(a^2+b^2)}\cos 2\alpha=0,\quad \frac{x}{a}\cos\beta+\&c.,\quad \frac{x}{a}\cos\gamma+\&c.,$$

where α, β, γ are the excentric angles of the corners of the triangle; (2) at the Lemoine-point, $x/a\cos 3\alpha=y/b\sin 3\alpha=(b^2-a^2)/2(a^2+b^2)$; and that its locus is the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=\frac{(a^2-b^2)^2}{4(a^2+b^2)^2}$; and (3) that the Brocard-points are given by the two systems

$$\frac{x/a\cos\alpha+y/b\sin\alpha+\frac{1}{2}}{(a^2\sin^2\beta+b^2\cos^2\beta)}=\frac{x/a\cos\beta+y/b\sin\beta+\frac{1}{2}}{(a^2\sin^2\gamma+b^2\cos^2\gamma)}=\frac{x/a\cos\gamma+y/b\sin\gamma+\frac{1}{2}}{(a^2\sin^2\alpha+b^2\cos^2\alpha)},$$

$$\frac{x/a\cos\alpha+y/b\sin\alpha-\frac{1}{2}}{(a^2\sin^2\gamma+b^2\cos^2\gamma)}=\frac{x/a\cos\beta+y/b\sin\beta-\frac{1}{2}}{(a^2\sin^2\alpha+b^2\cos^2\alpha)}=\frac{x/a\cos\gamma+y/b\sin\gamma-\frac{1}{2}}{(a^2\sin^2\beta+b^2\cos^2\beta)},$$

where $\cos\alpha+\cos\beta+\cos\gamma=0$, $\sin\alpha+\sin\beta+\sin\gamma=0$.

Solution by Professor ANDERSON, M.A.

Since the inscribed triangle is maximum, we have $\beta=\alpha+\frac{2}{3}\pi$ and $\gamma=\alpha+\frac{4}{3}\pi$. The centre of the ellipse is the centroid of the triangle, and hence, the sides being parallel to the tangents at the opposite angles, their equations are $x\cos\alpha/a+y\sin\alpha/b+\frac{1}{2}=0$, $x\cos\beta/a+\&c.$, $x\cos\gamma/a+\&c.$ To find the coordinates of the Lemoine-point we notice that, if p_1, p_2, p_3 be the lengths of the perpendiculars from the angles A, B, C on the opposite sides, and λ, μ, ν the lengths of the perpendiculars from the Lemoine-point, $\lambda p_1=\mu p_2=\nu p_3$.

Hence, if x, y be the coordinates of the point,

$$\frac{2(xb\cos\alpha+ya\sin\alpha)+ab}{b^2\cos^2\alpha+a^2\sin^2\alpha}=\frac{2(xb\cos\beta+ya\sin\beta)+ab}{b^2\cos^2\beta+a^2\sin^2\beta}$$
$$=\frac{2(xb\cos\gamma+ya\sin\gamma)+ab}{b^2\cos^2\gamma+a^2\sin^2\gamma}=\frac{2ab}{a^2+b^2};$$

$$\therefore\quad \frac{2x\sin(\beta-\alpha)+a(\sin\beta-\sin\alpha)}{(b^2\cos^2\alpha+a^2\sin^2\alpha)\sin\beta-(b^2\cos^2\beta+a^2\sin^2\beta)\sin\alpha}=\frac{2a}{a^2+b^2};$$

or
$$x=-a\cos 3\alpha\,(a^2-b^2)/2(a^2+b^2).$$

In the same way we find $y=-b\sin 3\alpha\,(a^2-b^2)/2(a^2+b^2)$.

Hence the locus of the Lemoine-point is the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=\frac{(a^2-b^2)^2}{4(a^2+b^2)^2}.$$

The equation of a line through the Lemoine-point parallel to the side BC may be written $x\cos\alpha/a+y\sin\alpha/b=k$; hence

$$k=-\frac{(a^2-b^2)}{2(a^2+b^2)}(\cos\alpha\cos 3\alpha+\sin\alpha\sin 3\alpha)=-\frac{a^2-b^2}{2(a^2+b^2)}\cos 2\alpha.$$

Thus the point is given by $\frac{x}{a}\cos\alpha+\frac{y}{b}\sin\alpha+\frac{a^2-b^2}{2(a^2+b^2)}\cos 2\alpha=0$, with two similar equations,

To find the Brocard-points we have

$$\frac{p_3}{p_2}\lambda = \frac{p_1}{p_3}\mu = \frac{p_2}{p_1}\nu, \quad \frac{p_2}{p_3}\lambda = \frac{p_3}{p_1}\mu = \frac{p_1}{p_3}\nu.$$

Hence, if x, y be the coordinates of a Brocard-point, we have, by substitution, the two systems of equations

$$\begin{aligned}&(x\cos\alpha/a + y\sin\alpha/b + \tfrac{1}{2})\,(b^2\cos^2\beta + a^2\sin^2\beta)\\ &= (x\cos\beta/a + y\sin\beta/b + \tfrac{1}{2})\,(b^2\cos^2\gamma + a^2\sin^2\gamma)\\ &= (x\cos\beta/a + y\sin\gamma/b + \tfrac{1}{2})\,(b^2\cos^2\alpha + a^2\sin^2\alpha).\end{aligned}$$

$$\begin{aligned}&(x\cos\alpha/a + y\sin\alpha/b + \tfrac{1}{2})\,(b^2\cos^2\gamma + a^2\sin^2\gamma)\\ &= (x\cos\beta/a + y\sin\beta/b + \tfrac{1}{2})\,(b^2\cos^2\alpha + a^2\sin^2\alpha)\\ &= (x\cos\gamma/a + y\sin\gamma/b + \tfrac{1}{2})\,(b^2\cos^2\beta + a^2\sin^2\beta).\end{aligned}$$

9148. (Professor Asutosh Mukhopadhyay.) — Prove that (1) the reciprocal polar of the evolute of the hypocycloid

$$\left(\frac{x}{a}\right)^{\frac{2}{3}} + \left(\frac{y}{b}\right)^{\frac{2}{3}} = 1 \quad \text{is} \quad \left(\frac{x^2}{a^2} + \frac{y^2}{b^2}\right)^3 = \left(\frac{x^4}{a^2} - \frac{y^4}{b^2}\right)^2,$$

the radius of inversion being unity; and (2), if $a = b$, the polar equation of the reciprocal polar is $r = a\sec 2\theta$.

Solution by Professor Schoute.

The hypocycloid is given by $x = a\cos^3\phi$, $y = b\sin^3\phi$; and the normal in the point ϕ is $-ax\cos\phi + by\sin\phi = b^2\sin^4\phi - a^2\cos^4\phi$. When this equation represents the polar $xx' + yy' = 1$ of (x', y') with respect to the circle $x^2 + y^2 = 1$, we have

$$x' = (-a\cos\phi)/(b^2\sin^4\phi - a^2\cos^4\phi), \quad y' = (b\sin\phi)/(b^2\sin^4\phi - a^2\cos^4\phi),$$

and by elimination of ϕ, as $\mathrm{Tg}\,\phi = -\dfrac{ay'}{bx'}$, for the reciprocal polar of the evolute

$$\left(\frac{y^4}{b^2} - \frac{x^4}{a^2}\right)^2 = \left(\frac{x^2}{a^2} + \frac{y^2}{b^2}\right)^3,$$

a curve of the 8th order of which the origin is a sextuple point, &c.

10724. (Professor Anderson, M.A.)—Perpendiculars are drawn to the symmedians of a triangle, at its angular points, forming another triangle. Show that the circumcentre of the former is the centroid of the latter.

Solution by R. Tucker, M.A.

Let AK, BK, CK be the symmedians, and A′B′C′ the new triangle: then, since $c\beta - b\gamma = 0$ is the equation to AK, the equation to B′C′ is

$(b + c\cos A)\,\beta + (c + b\cos A)\,\gamma = 0$; hence the points A', B', C' are

$$\frac{-a}{b + a\cos C \,.\, c + a\cos B} = \frac{\beta}{c + a\cos B \,.\, a + b\cos C} = \frac{\gamma}{a + c\cos B \,.\, b + a\cos C}$$

$$\text{&c.} \quad = \quad \text{&c.} \quad = \frac{2\Delta}{2abc\,(1 + \cos A\cos B\cos C'}$$

whence the centroid of A'B'C' is readily seen to be $R\cos A$, $R\cos B$, $R\cos C$; therefore, &c.

10666. (Professor ORCHARD, M.A.)—A triangle, PQR, right-angled at Q, is inscribed in the conic $4x^2 - 9y^2 - 36 = 0$; show that the envelope of the line PR is the conic $25\,(4x^2 - 9y^2) = 6084$.

Solution by Professors ANDERSON ; AIYAR ; *and others.*

Let the coordinates of the point Q be α, β, and the equation of the line PR $lx + my + 1 = 0$. Then, writing down the equation of the pair of lines PQ, RQ, we find easily the condition of perpendicularity to be $13l\alpha - 13m\beta + 5 = 0$.

The line PR, therefore, passes through the point whose coordinates are $\frac{1}{5}^3\alpha$, $-\frac{1}{5}^3\beta$.

The locus of this point is the conic stated in the Question.

6810. (J. R. HARRIS, M.A.)—A chain is lying in a given curve on a smooth table, and an impulse is communicated to it along the tangent at one end. Apply the theorem that the kinetic energy of the resulting motion is a minimum to determine the impulsive tension at any point of the chain. If the energy of the resulting motion be distributed uniformly along the chain, and be λ per unit of length, prove that the tension and the mass per unit of length are given by

$$T = \sqrt{(\tfrac{1}{2}\lambda\mu\rho)}\,[\sqrt{(\rho + 2s)} - \sqrt{(\rho - 2s)}]\,;\ \mu\rho \propto \exp.\int \frac{1}{\rho}\,d\,(\rho^2 - 4s)^{\frac{1}{2}}\,;$$

ρ being the radius of curvature of the chain at a point whose distance from the free end is s. [exp. $x \equiv e^x$.]

Solution by D. EDWARDES, B.A. ; G. EASTWOOD, M.A. ; *and others.*

We have $$\frac{d^2T}{ds^2} - \frac{1}{\mu}\frac{d\mu}{ds}\frac{dT}{ds} - \frac{T}{\rho^2} = 0 \qquad \text{(1)}.$$

If now we put $\mu(\dot{x}^2 + \dot{y}^2) = 2\lambda$, the equations of motion, s being the independent variable, give

$$2\mu\lambda = \left(\frac{dT}{ds}\right)^2 + \frac{T^2}{\rho^2} \qquad \text{(2)}.$$

(1) × T + (2) integrated gives

$$T\frac{dT}{ds} = 2\mu\lambda s,$$

s being measured from the free extremity. Eliminating dT/ds from this equation and (2), we get

$$(T^2-\mu\lambda\rho^2)^2 = \mu^2\lambda^2\rho^2(\rho^2-4s^2) \quad \ldots\ldots (3).$$

Again, differentiating (2) and combining the result with (1), there results

$$\frac{dT^2}{ds} - T^2\frac{d}{ds}\log(\mu\rho) + \lambda\rho^2\frac{d\mu}{ds} = 0;$$

whence, integrating, and writing

$$\int_0^s \frac{1}{\mu}\frac{d(\mu\rho)}{ds}\,ds - \rho + \rho_0 \text{ for } \int_0^s \frac{\rho}{\mu}\frac{d\mu}{ds}\,ds,$$

$$T^2-\mu\lambda\rho^2 = -\mu\lambda\rho\int_0^s \frac{1}{\mu}\cdot\frac{d(\mu\rho)}{ds}\,ds - \mu\lambda\rho_0\rho \quad \ldots\ldots (4).$$

Returning now to (3), this gives

$$T^2 = \tfrac{1}{2}\mu\lambda\rho\,[2\rho - 2(\rho^2-4s^2)^{\frac{1}{2}}],$$

since T vanishes with s, or

$$T = \sqrt{(\tfrac{1}{2}\mu\lambda\rho)}\,[(\rho+2s)^{\frac{1}{2}} - (\rho-2s)^{\frac{1}{2}}].$$

Then, from (3), (4),

$$(\rho^2-4s^2) = \int_0^s \frac{1}{\mu}\frac{d(\mu\rho)}{ds}\,ds + \mu\lambda\rho_0,$$

where ρ is the radius of curvature at the free end, or

$$d(\rho^2-4s^2)^{\frac{1}{2}} = \frac{1}{\mu}\,d(\mu\rho);$$

therefore

$$\log(\mu\rho) = \int \frac{1}{\rho}\,d(\rho^2-4s^2)^{\frac{1}{2}}.$$

10678. (J. O'Byrne Croke, M.A.)—Show that, if Δ be the discriminant of the conic $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$, and $R^2 = 4h^2 + (a-b)^2$, the semi-axes α, β are given by

$$\alpha^2 = 2\Delta/\{(h^2-ab)(a+b-R)\}, \quad \beta^2 = 2\Delta/\{(h^2-ab)(a+b+R)\}.$$

Solution by Professors Anderson; Zerr; *and others.*

The equations of the conic referred to axes through its centre parallel to the axes of coordinates, and for determining the lengths of the semi-axes, are

$$ax^2 + 2hxy + by^2 - \Delta/(h^2-ab) = 0,$$

$$(h^2-ab)^2 r^4 + (a+b)\Delta r^2 - \Delta^2/(h^2-ab) = 0;$$

$$\therefore\ r^4 + \frac{(a+b)\Delta}{(h^2-ab)^2} - \frac{\Delta^2}{(h^2-ab)^3} = 0; \text{ also } (a+b)^2 - R^2 = -4(h^2-ab).$$

Hence the equation becomes

$$r^4 - \frac{2\Delta}{h^2-ab}\left(\frac{1}{a+b-R}+\frac{1}{a+b+R}\right) + \frac{4\Delta^2}{(h^2-ab)^2\{(a-b)^2-R^2\}} = 0,$$

$$\text{or } \left\{r^2 - \frac{2\Delta}{(h^2-ab)(a+b-R)}\right\}\left\{r^2 - \frac{2\Delta}{(h^2-ab)(a+b+R)}\right\} = 0;$$

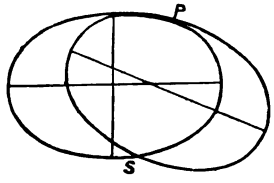
hence the lengths of the semi-axes are those given in the question.

10240. (R. W. D. Christie.)—Prove that any power of any prime number equals either the sum or the difference of two squares.

Solution by J. J. Barniville; H. J. Woodall; *and others.*

$(n^a+1)^2-(n^a-1)^2 = 4n^a$; hence $n^a = \left\{\frac{1}{2}(n^a+1)\right\}^2 - \left\{\frac{1}{2}(n^a-1)\right\}^2$, and, since n is prime, $\frac{1}{2}(n^a \pm 1)$ is a whole number. [$n = 2$ fails.]

Note on Question 10286. *By the* Editor.



The diagram and solution given on p. 85 of Vol. liii. are both wrong. As to the figure, if the conics osculate—that is to say, have three-point contact—in P, they must meet in a fourth point S.

Moreover, both diagram and solution assume the conics to be concentric.

Professor Wolstenholme "doubts if the theorem be true, but has no time or inclination to find out." We shall be glad to have an investigation from some other contributor.

8669. (Alpha.)—Any point P being taken on a rectangular hyperbola and QQ′ being any diameter, circles are drawn touching the hyperbola at P and passing through Q, Q′ respectively; prove that these circles will be equal.

Solution by R. Knowles, B.A.; Rev. T. Galliers, M.A.; *and others.*

Let the equation to the hyperbola be $xy = a^2$, coordinates of Q, Q′, a', b'; $-a'$, $-b'$ respectively; let the equation to the circle be

$$x^2+y^2+Dx+Ey+F = 0,$$

putting in this $x = a^2/y$, we obtain $y^4+Ey^3+Fy^2+a^2Dy+a^4 = 0$, but this is also equal to $(y^2-b'^2)(y-y')^2$; y' being the ordinate of P; equating

these results, we have $D = -\frac{2a^2}{y'}$, $E = -2y'$, $F = \frac{a^4 + y'^4}{y'^2}$; and the equation to the circle is $x^2 + y^2 - \frac{2a^2}{y'}x - 2y'y + \frac{a^4 + y'^4}{y'^2} = 0$. This equation represents the point P, which may be considered as a circle with an indefinitely small radius; and the circles drawn as in the question may in this sense be considered equal.

10732. (J. D. H. Dickson, M.A. Arising out of 10690.)—Find integral values of x, y, z to satisfy the equations

$$x + y + z = 0, \quad x^2 + y^2 + z^2 = 2\square.$$

Solution by D. Biddle; Rev. J. L. Kitchin; *and others.*

$z = -(x+y)$, therefore $x^2 + y^2 + xy = \square$, and $(x+y)^2 - xy = (x+y-n)^2$; whence $y = 2n + 3n^2/(x-2n)$. It is now clear that, n being any integer, x and y are both integral when $x - 2n = 1$, or when $x - 2n = 3n^2$, or when $x - 2n = 3n$, or when $x - 2n = n$ (or any other factor of $3n^2$). Consequently, x and y may be interchangeably $2n+1$ and $3n^2 + 2n$, or $5n$ and $3n$. Thus, if $n = 1$, we have, from either pair, $x = 3$, $y = 5$, $z = -8$, $x^2 + y^2 + z^2 = 2 \cdot 7^2$. If $n = 2$, we have, from the first pair, $x = 5$, $y = 16$, $z = -21$, $x^2 + y^2 + z^2 = 2 \cdot 19^2$; and, from the second pair, $x = 10$, $y = 6$, $z = -16$, $x^2 + y^2 + z^2 = 2 \cdot 14^2$. If $n = 3$, we have, from the first pair, $x = 7$, $y = 33$, $z = -40$, $x^2 + y^2 + z^2 = 2 \cdot 37^2$; and, from the second pair, $x = 15$, $y = 9$, $z = -24$, $x^2 + y^2 + z^2 = 2 \cdot 21^2$, and so on.

10577. (H. J. Woodall.)—A certain complex quantity has the operation $(\ldots)^2$ repeated on it a number of times (producing the series, say, k, k^2, k^4, k^8, $\ldots k^{2^n}$, &c.). If the values recur, in a period of q, required an investigation of the form of the complex.

Solution by Professors Anderson; Sarkar; *and others.*

We have $k^{2^n} = k^{2^{(n+q)}}$ for all positive integral values of n, including zero; hence $k^{2^n}\left(1 - k^{2^{(n+q)} - 2^n}\right) = 0$; hence $k^{(2^q - 1)2^n} = 1$, or $k = 0$. The former equation gives $k^{2^q - 1} = 1$; hence

$$k = \cos\frac{2S\pi}{2^q - 1} + (-1)^{\frac{1}{2}}\sin\frac{2S\pi}{2^q - 1}, \text{ or } 0,$$

where S is any integer positive or negative, or zero.

10023. (P. Brasseur.)—Démontrer que tout nombre composé de 3^n chiffres 1 est un multiple de 3^n.

Solution by Professor Beyens; J. C. St. Clair; *and others.*

Si la proprieté signalée est verifiée pour 3^{n-1} chiffres 1, elle le sera pour 3^n, car on aura :—

Un nombre composé de 3^n chiffres 1

$$= \overline{111111\ldots111}^{3^{n-1}}, \overline{111111\ldots111}^{3^{n-1}}, \overline{111111\ldots111}^{3^{n-1}}$$

$$= m \,.\, 3^{n-1} \,.\, 10^{2.3^{n-1}} + m \,.\, 3^{n-1} \,.\, 10^{3^{n-1}} + m \,.\, 3^{n-1}$$

$$= m \,.\, 3^{n-1} (10^{2.3^{n-1}} + 10^{3^{n-1}} + 1);$$

mais $10^{3^{n-1}} = (9 \,.\, 111111\ldots111)^{3^{n-1}} + 1$, ou $10^{3^n-1} = 9 \,.\, m \,.\, 3^{n-1} + 1$,

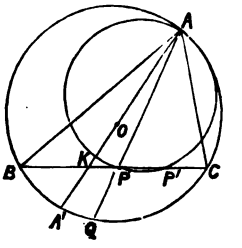
et par suite $$10^{2.3^{n-1}} = m \,.\, 3^{n-1} + 1,$$

donc $$10^{2.3^{n-1}} + 10^{3^{n-1}} + 1 = m \,.\, 3,$$

et par suite le nombre de 3^n chiffres 1 est multiple de $3 \,.\, 3^{n-1} = 3^n$, et comme le theorème est certaine pour 3, 3^2, 3^3, il est général.

10156. (The Editor.)—On the base BC of a triangle find a point P, such that AP^2 may have to BP . PC a given ratio.

Solution by Professor Wolstenholme, M.A., Sc.D.



Let O be the circumcentre, AA′ a diameter of the circumcircle; divide AA′ in K, so that $AK = m \,.\, KA'$, then the circle on AK as diameter will meet BC in two points P, such that

$$AP^2 = m \,.\, BP \,.\, PC.$$

Let AP meet the circumcircle in Q, then

$$AP^2 : BP \,.\, PC = AP^2 : AP \,.\, PQ$$
$$= AP : PQ = AK : KA' = m : 1.$$

If m be negative, K will lie outside AA′, and the two points P will then lie in the production of BC, and $AP^2 = m \,.\, BP \,.\, PC$, m being negative and BP, PC of opposite signs. When m is positive, it must not be $< (b+c)^2 - a^2 : a^2$, or the circle will not meet BC in real points, and there is a corresponding limit $(b-c)^2 - a^2 : a^2$ when m is negative and $m+1$ positive. When m has either of these limiting values, the circle *touches* BC in P and AP bisects the angle A or its supplement. [See also Vol. liii., pp. 31, 2.]

9916. (R. KNOWLES, B.A.)—A rectangular hyperbola touches a given parabola at P, and cuts it in one other point Q; prove that (1) the locus of its centre is a circle which (*a*) has its centre on the directrix; (*b*) has its diameter equal to the radius of curvature at P; (*c*) intersects the circle of curvature at P on the tangent at P; (2) the locus of the mid-point of PQ is a parabola, which touches the given parabola at P.

Solution by R. WHAPHAM, B.A.; Rev. T. GALLIERS; *and others.*

The rectangular hyperbola osculating $y^2-4ax=0$ at P $(x'y')$, is

$$2amx^2-(my'+2a)\,xy-2amy^2+2a\,(4am+3y')\,x$$
$$-(6ax'-mx'y')\,y+2a\,(y'-mx')\,x'=0;$$

equations for the centre are

$$4amx-(my'+2a)\,y+2a\,(4am+3y')=0 \quad \text{..... (i.)},$$
$$(my'+2a)\,x+4amy+6ax'-mx'y'=0 \quad \text{..... (ii.)}.$$

Eliminating m from (i.) and (ii.), we get for the locus of the centre

$$4ax^2+4ay^2+8a^2x-4y'\,(x'+3a)\,y+3y'^2\,(2a+x')=0 \quad \text{.....(iii.)}.$$

The x coordinate of the centre of this circle is $-a$; therefore

the centre of the locus is on the directrix of the parabola.....(*a*),

$$(\text{radius})^2 \text{ of (iii.)} = \frac{16a^4+4y'^2\,(x'+3a)^2-12ay'^2\,(2a+x')}{16a^2} = \frac{a\,(a+x')^3}{a^2}.$$

Now radius of curvature at $x'y'$ is $\frac{2\sqrt{a}\,(a+x')^{\frac{3}{2}}}{a}$; therefore result (*b*).

Circle of curvature at $x'y'$ is

$$ax^2+ay^2-2a\,(2a+3x')\,x+2x'y'y-3ax'^2=0 \quad \text{..... (iv.)};$$

where (iii.), (iv.) intersect we have, by subtraction,

$$24a^2x+24ax'x-(12x'y'+12ay')\,y+24a^2x'+24ax'^2=0,$$

or

$$yy'=2a\,(x+x');$$

therefore (iii.) and (iv.) intersect on the tangent at P ..... (*c*).

Again, let α, β be coordinates of mid-point of PQ; Q being $x''y''$, where $y-mx-(y'-mx')=0$ meets $y^2-4ax=0$, we have

$$m^2x^2-2\left\{2a-m\,(y'-mx')\right\}x+(y'-mx')^2=0;$$

whence

$$\alpha=\frac{x'+x''}{2}=\frac{2a-m\,(y'-mx')}{m^2} \quad \text{.....(v.)},$$

$$\beta=\frac{y'+y''}{2}=\frac{2a}{m} \quad \text{..... (vi.)}.$$

Eliminating m, we get for the locus of (α, β), $y^2-yy'=2ax-2a\alpha'$, a

parabola. The tangent at $x'y'$ to this parabola is

$$2yy' - (y+y')\,y' = 2a\,(x+x') - 4ax', \quad \text{or} \quad yy' = 2a\,(x+x');$$

therefore locus of mid-point of PQ is a parabola touching the given parabola at P.

10791. (Rev. T. C. Simmons, M.A.)—Prove that the mean value of the nth power of the distance between two points taken at random within a given circle, n being not less than -1, is

$$\frac{2^{n+4}}{(n+2)(n+3)\,\pi^{\frac{1}{2}}} \cdot \frac{\Gamma\left[\frac{1}{2}(n+5)\right]}{\Gamma\left[\frac{1}{2}(n+6)\right]}\, r^n$$

Solution by Professor Zerr, M.A.

The mean value of the nth power of the distance of a point on the circumference of a circle from all points inside the circle is

$$\mathrm{M}_0 = \frac{\iint r^n\, d\mathrm{S}}{\pi a^2} = \frac{2}{\pi a^2}\int_0^{\frac{1}{2}\pi}\int_0^{2a\cos\theta} r^{n+1}\, d\theta\, dr = \frac{2^{n+2}}{(n+2)\,\pi^{\frac{1}{2}}} \cdot \frac{\Gamma\left[\frac{1}{2}(n+3)\right]}{\Gamma\left[\frac{1}{2}(n+4)\right]}\, a^n.$$

By Art. 241, p. 352, Ex. 1, Williamson's *Int. Cal.*, we have for the required mean $\quad d\left\{(\pi r^2)^2\,\mathrm{M}\right\} = 4\pi^2 \mathrm{M}_0 r^3 dr\,;$

therefore $\quad \pi^2 r^4 \mathrm{M} = \dfrac{2^{n+4}\pi^2}{(n+2)\,\pi^{\frac{1}{2}}} \cdot \dfrac{\Gamma\left[\frac{1}{2}(n+3)\right]}{\Gamma\left[\frac{1}{2}(n+4)\right]} \int_0^r r^{n+3}\, dr\,;$

therefore

$$\mathrm{M} = \frac{2^{n+4}}{(n+2)(n+4)\,\pi^{\frac{1}{2}}} \cdot \frac{\Gamma\left[\frac{1}{2}(n+3)\right]}{\Gamma\left[\frac{1}{2}(n+4)\right]}\, r^n = \frac{2^{n+4}}{(n+2)(n+3)\,\pi^{\frac{1}{2}}} \cdot \frac{\Gamma\left[\frac{1}{2}(n+5)\right]}{\Gamma\left[\frac{1}{2}(n+6)\right]}\, r^n.$$

10691. (Professor Hudson, M.A.)—If a face of Cleopatra's needle be a trapezoid having its two parallel sides of lengths 5 ft. 4 in. and 7 ft. 11 in., and their distance apart 61 ft. 6 in., and if it be exposed to a uniform normal pressure of 40 lbs. to the square foot, show that the moment of this pressure about the side 7 ft. 11 in. is $468578\frac{3}{4}$ foot-pound units.

Solution by Professor Zerr, M.A.

The pressure at the centre of gravity, if it were all transmitted to that point, is $\frac{5\cdot3}{8} \times \frac{1\cdot2\cdot3}{2} \times 40 = 16297\frac{1}{2}$ lbs. The distance of the side 7 ft. 11 in. from the C. of G. is $25\frac{1\cdot2\cdot3}{3\cdot1\cdot8}$ ft. The moment

$$= 16297\tfrac{1}{2} \text{ lbs.} \times 25\tfrac{1\cdot2\cdot3}{3\cdot1\cdot8} \text{ ft.} = 468578\tfrac{3\cdot1}{10\cdot1} \text{ foot-pounds.}$$

[Distance of C. G. from side 7 ft. 11 in. is $\dfrac{10.8+7.11}{5.4+7.11}$ of $\dfrac{61.6}{3}$ feet $= \dfrac{223}{159} \times \dfrac{41}{2}$ feet; and the moment $= \dfrac{53}{8} \times \dfrac{123}{2} \times 40 \times \dfrac{223}{159} \times \dfrac{41}{2}$ = result.]

10068. (Professor DE LONGCHAMPS.)—Résoudre l'équation

$$(x+b+c)(x+c+a)(x+a+b)-abcx/(a+b+c)=0.$$

Solution by Professor BEYENS; Rev. J. L. KITCHIN, M.A.; *and others.*

L'équation peut se mettre sous la forme

$$x(x+a+b+c)^2+\frac{(a+b)(a+c)(b+c)x}{a+b+c}+(a+b)(a+c)(b+c)=0,$$

ou $$(x+a+b+c)\left\{x(x+a+b+c)+\frac{(a+b)(a+c)(b+c)}{a+b+c}\right\}=0;$$

donc, &c.

10608. (W. J. GREENSTREET, M.A.)—If the normal at P on the curve $r^2=a^2\cos 2\theta$ meet the axis in H; and if HQ, HR be drawn perpendicular to OH, OP, meeting the circle on the join of the poles of the curves as diameter in Q, R; prove that $HQ : OQ = RP^2 : OQ^2$.

Solution by Professors ANDERSON, MUKHOPADHYAY, *and others.*

We have $OQ = a/2^{\frac{1}{2}}$, and $\angle OPH = 2\theta$; hence, if HR meet OP in N,

$$RP^2=\tfrac{1}{2}a^2+(PN)^2-(ON)^2=\tfrac{1}{2}a^2+(PN+ON)(PN-ON)$$
$$=\tfrac{1}{2}a^2-r^2(\sin 2\theta\cos\theta-\sin\theta\cos 2\theta)/\sin 3\theta,$$

which reduces to $a^2\sin\theta/2\sin 3\theta$.

Again, $(HQ)^2=\tfrac{1}{2}a^2-(OH)^2=\tfrac{1}{2}a^2-r^2\sin^2 2\theta/\sin^2 3\theta,$

or $a^2\sin^2\theta/2\sin^2 3\theta$;

therefore $HQ = a\sin\theta/2\sin 3\theta$; therefore, &c.

10135. (R. W. D. CHRISTIE.)—Prove that

$$\sqrt{n}=2^{\frac{1}{2}(n-1)}\cos a\cos 3a\cos 5a\ldots\cos\tfrac{1}{2}(n-1).a,$$

when n is odd, and $a=\pi/2n$.

Solution by J. J. BARNIVILLE; Rev. J. L. KITCHIN; *and others.*

$$\frac{\sin n\phi}{\sin\phi}=2^{n-1}\sin(2a+\phi)\sin(4a+\phi)\ldots\sin(2na-2a+\phi);$$

now let ϕ diminish without limit; then, since the limit of $\frac{\sin n\phi}{\sin\phi}$ is n, we obtain

$$n=2^{n-1}\sin 2a\sin 4a\sin 6a\ldots\sin 2a(n-1);$$

therefore

$$\sqrt{n}=2^{\frac{1}{2}(n-1)}\sqrt{\sin 2a\sin 4a\ldots}$$
$$=2^{\frac{1}{2}(n-1)}\cos a\cos 3a\cos 5a\ldots\cos\tfrac{1}{2}(n-1)a.$$

10375. (Capitaine DE ROCQUIGNY.)—Quel que soit n, démontrer que l'un des deux nombres $3^{3n}+2^{3n}$, $3^{3n}-2^{3n}$ est divisible par 35.

Solution by W. J. GREENSTREET, M.A.; R. KNOWLES, B.A.; *and others.*

$3^{3n} \pm 2^{3n} = 27^n \pm 8^n = \text{M}(35)$, $\pm$ as n is odd or even.

10174. (F. BOUVIER.)—Dans un hexagone inscrit, les six circonférences passant par le centre du cercle circonscrit et par les extrémités de chaque côté se coupent deux à deux en trois points situés sur une circonférence passant par le centre du cercle circonscrit.

Solution by J. C. ST. CLAIR; M. BRIERLEY; *and others.*

The (opposite) sides of an inscribed hexagon meet two and two in three points on the Pascal line. If we invert the figure with respect to the circumcircle, the sides of the hexagon, as well as the Pascal line, become circles passing through the circumcentre; which proves the proposition.

9030. (Professor IGNACIO BEYENS. Suggérée par la Quest. 8326.)—Si du centre de gravité on abaisse perpendiculaires GL, GL', GL'' sur les hauteurs d'un triangle ABC, et si de l'orthocentre O on abaisse des perpendiculaires sur les médianes, OM, OM', OM''; les deux triangles LL'L'', MM'M'' sont *homologiques* et le centre d'*homologie* est le point K de Lemoine du triangle ABC.

Solution by the PROPOSER.

Dans la question (8326) il est demontré que les lignes droites LM, L'M', L''M'' vont à passer par le point K: donc le théorème est demontré.

10776. (R. TUCKER, M.A.)—P, Q, R are three co-normal points on a parabola. The tangents at these points cut any double ordinate in L, M, N; the other tangents through L, M, N intersect in p, q, r. Prove that the circle pqr passes through the vertex.

Solution by the PROPOSER; R. KNOWLES, B.A.; *and others.*

The tangent at $P(am_1^2, 2am_1)$ is $x - m_1 y + am_1^2 = 0$; hence, if the ordinate is $x = ka$, L is the point $[ka, a(k+m_1^2)/m_1]$. The second tan-

gent through L is $m_1^2 x - k m_1 y + k^2 a = 0$, and the coordinates of r are

$$(k^2 a m_3/\mu, \quad -k a m_3^2/\mu).$$

$$[S_1 \equiv m_1 + m_2 + m_3 = 0, \quad S_2 \equiv \Sigma m^2, \quad m_1 m_2 m_3 \equiv \mu.]$$

The equation to the tangent-circle pqr is

$$x^2 + y^2 - ax + ka\,[2k^2 + S_2]\,y/2\mu = 0.$$

10727. (Professor WOLSTENHOLME, M.A., Sc.D.)—If the equation of a conic be $u \equiv (a, b, c, f, g, h \mathbin{\textstyle\sum} x, y, 1)^2 = 0$,

the equation $4\lambda u = \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2$, where $(\lambda - a)(\lambda - b) = h^2$ is the equation of its directrices.

Solution by Professor RAMASWAMI AIYAR.

If $u \equiv \lambda (x-p)^2 + \lambda (y-q)^2 - (lx + my + 1)^2$, by comparison we get at once $(\lambda - a)(\lambda - b) = h^2$. Again, the directrix $lx + my + 1 = 0$ is such that points upon it satisfy an equation of the form $u = \lambda (x-p)^2 + \lambda (y-q)^2$, therefore $\frac{\partial u}{\partial x} = 2\lambda (x-p)$; $\frac{\partial u}{\partial y} = 2\lambda (y-q)$.

Eliminating p and q, we obtain the stated equation of the directrices.

The equation for λ might have been obtained in the other form by eliminating l, m from $\frac{\partial^2 u}{\partial x^2}$, $\frac{\partial^2 u}{\partial x \partial y}$, and $\frac{\partial^2 u}{\partial y^2}$.

[One value of λ gives the real, the other the impossible, directrices. The Question may be otherwise stated thus:—The equation of a pair of directrices of any conic $u = 0$, is $2\lambda u = \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2$, where λ is given by the equation $\left(\lambda - \frac{\partial^2 v}{\partial x^2}\right)\left(\lambda - \frac{\partial^2 u}{\partial y^2}\right) = \left(\frac{\partial^2 u}{\partial x \partial y}\right)^2$. The greater root to be taken when Δ is negative.]

10762. (JOSEPH BLATER.) — Prove that, if the first five decimal places of the reciprocal of any number be given, the next five places may readily be found by the use of a Table of Quarter Squares.

Solution by the PROPOSER.

Let the 5-place reciprocal be denoted by r or r_1 according as it is taken to be less than or greater than the proper value. If z is the given number,

$zr = 10^{2n} - d$ and $zr_1 = 10^{2n} + d_1$, where d denotes the difference. For a 10-place reciprocal R we obtain

$$R = r\left(1 + \frac{d}{10^{10}}\right) \text{ if } zr = 10^{10} - d = r_1\left(1 - \frac{d_1}{10^{10}}\right) \text{ if } zr_1 = 10^{10} + d_1.$$

To find an unlimited number of places of the reciprocal, we get (according as we suppose $zr = 10^{2n} - d$ or $zr_1 = 10^{2n} + d_1$),

$$R = r\left(1 + \frac{d}{10^{2n}} + \frac{d^2}{10^{4n}} + \frac{d^3}{10^{6n}} \ldots\right) \text{ or } R = r_1\left(1 - \frac{d_1}{10^{2n}} + \frac{d_1^2}{10^{4n}} - \ldots\right).$$

Beyond 10 places of the reciprocal the Table of Quarter Squares offers no advantage over other methods, but the "Erleichterungstafel" (Trübner & Co.) from this point onwards begins to be of use and more than fulfils our requirements.

Example: $z = 114591559$.

Take the case $r = 87266$.

$$\text{Factors} \begin{cases} 87266 & 87266 \\ 114590000 & + \quad 1559 \end{cases} \begin{matrix} 88825 = 1972470^* \ldots \\ 85707 = 1836422^* \ldots \end{matrix}$$

$$98725 = 2436,656,406\}^* \qquad 136048$$

$$75807 = 1436,675,312\}^* \quad = \quad 999981094$$

$$zr = 10^{10} - 53012 \quad = \quad 9999946988$$

$$1 + \frac{d}{10^{10}} = 1000053012$$

$$r = \quad 87266$$

$$r\frac{d}{10^{10}} = \quad \begin{cases} 140278 = 49185 \ldots \\ 34252 = \quad 2933 \ldots \end{cases} = 46262$$

$$+ r = \quad 87266$$

$$r\left(1 + \frac{d}{10^{10}}\right) = R = 8726646262$$

3768. (J. B. Sanders.)—A conical vessel of base-radius r, and height h, is filled with fluid; find the time in which the fluid will descend through half the height, if an orifice be made at the vertex of the cone, when the axis is vertical.

Solution by Professors Zerr, Beyens, *and others.*

Using the equations

$$t = -\pi \int \frac{y^2\,dx}{k\,(2gx)^{\frac{1}{2}}}, \text{ and } y^2 = x^2 \tan^2 a,$$

we have

$$t = \frac{\pi r^2}{20k}\left\{2^{\frac{1}{2}} - 1\right\}\left(\frac{h}{g}\right)^{\frac{1}{2}};$$

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* Tabular numbers.

hence time for emptying is

$$t = -\frac{\pi \tan^2 \alpha}{k\,(2g)^{\frac{1}{2}}} \int_0^h x^{\frac{3}{2}}\,dx = \frac{2\pi r^2}{5k}\left(\frac{h}{2g}\right)^{\frac{1}{2}}.$$

9204. (Professor Byomakesa Chakravarti, M.A.)—Show that, if k be the radius of inversion, the reciprocal polar of the evolute of

$$(x/a)^m + (y/b)^m = 1 \quad \text{is} \quad k^2\left\{(+x/a)^{m/(m-1)} + (-y/b)^{m/(m-1)}\right\}^{1/m}$$
$$= (b/y)(+x/a)^{1/(m-1)} + (ax)(-y/b)^{1/(m-1)}.$$

Solution by A. Kahn, M.A.; Sarah Marks; *and others.*

The point given by

$$x = a\,(\cos\phi)^{2/m}, \quad y = b\,(\sin\phi)^{2/m}$$

is a point on the curve $(x/a)^m + (y/b)^m = 1$ .....(1).

Now $dy/dx = -b/a\,(\cot\phi)^{2(1-1/m)}$;

hence the equation to the normal of (1), *i.e.*, to the tangent of the evolute of (1), at the point (ϕ) is

$$y\,.\,b\,(\cos\phi)^{2(1-1/m)} - x - a\,(\sin\phi)^{2(1-1/m)}$$
$$= b^2\,(\sin\phi)^{2/m}\,(\cos\phi)^{2(1-1/m)} - a^2\,(\cos\phi)^{2/m}\,(\sin\phi)^{2(1-1/m)} \ldots\ldots (2);$$

$$\therefore \quad p^2 = \frac{\left[b^2\,(\sin\phi)^{2/m}\,(\cos\phi)^{2(1-1/m)} - a^2\,(\cos\phi)^{2/m}\,(\sin\phi)^{2(1-1/m)}\right]^2}{b^2\,(\cos\phi)^{4(1-1/m)} + a^2\,(\sin\phi)^{4(1-1/m)}},$$

where p is the length of the perpendicular distance from the centre in (1); therefore, if x, y be the point on the reciprocal polar corresponding to this value of p, we have

$$(x^2+y^2)\left[b^2\,(\sin\phi)^{2/m}\,(\cos\phi)^{2(1-1/m)} - a^2\,(\cos\phi)^{2/m}\,(\sin\phi)^{2(1-1/m)}\right]^2$$
$$= k^4\left[b^2\,(\cos\phi)^{4(1-1/m)} + a^2\,(\sin\phi)^{4(1-1/m)}\right] \ldots\ldots (3).$$

Again, (x, y) is on the straight line $y/x = -b/a\,(\cot\phi)^{2(1-1/m)}$; therefore we have $(\cot\phi)^{2(1-1/m)} = -ay/bx$ ..... (4).

Eliminating ϕ between (3), (4), we obtain the result.

10663. (Professor Villademoros.)—Deux triangles ABC, AB'C' ont le sommet A et la médiane AM communs. Démontrer que, si le côté BC est parallèle à la bissectrice de l'angle B'AC', et tel que $MB^2 = AB'.AC'$, réciproquement le côté B'C' sera parallèle à la bissectrice de l'angle BAC, et tel que $MB'^2 = AB : AC$.

Solution by Professors ANDERSON, CHAKRIVARTI, *and others.*

Let the bisector of the angle BAC meet BC in D, and let the sides of the triangle AMD, AM, MD, AD be denoted by x, y, z. If p be the sum of the sides AB, AC, we have

$$p^2 = 2(y^2-z^2+x^2)(y^2-z^2-x^2)/(y^2+z^2-x^2).$$

If y and z be interchanged, the value of p is unaltered, showing that

$$AB' + AC' = AB + AC.$$

Now $(AB)^2 + (AC)^2 = 2(BM)^2 + 2AM^2 = 2AB'.AC' + 2AM^2$;

also $AB'^2 + AC'^2 = 2B'M^2 + 2AM^2$;

therefore $2B'M^2 = (AB+AC)^2 - AB^2 - AC^2 = 2AB.AC.$

10786. (J. BRILL, M.A.)—The moments of n coplanar forces about a point O in their plane are $m_1, m_2, \ldots, m_n$. Through O a straight line is drawn meeting the lines of action of the forces in the points $r_1, r_2, \ldots, r_n$, and a point R is taken on this line so that

$$\frac{m_1+m_2+\ldots+m_n}{OR} = \frac{m_1}{Or_1} + \frac{m_2}{Or_2} + \ldots + \frac{m_n}{Or_n};$$

prove that the locus of R is the line of action of the resultant of the given forces.

Solution by Professor ZERR; T. CLARKE; *and others.*

Let the equations of the lines of action of the forces be

$$\rho\cos(\theta-\alpha_1) = p_1, \quad \rho\cos(\theta-\alpha_2) = p_2, \text{ \&c.};$$

then $$\frac{\Sigma(m)}{\rho} = \Sigma\left(\frac{m}{p}\cos\alpha\right)\cos\theta + \Sigma\left(\frac{m}{p}\sin\alpha\right)\sin\theta,$$

or $$\Sigma\left(\frac{m}{p}\cos\alpha\right)x + \Sigma\left(\frac{m}{p}\sin\alpha\right)y = \Sigma(m).$$

This is plainly the equation of the line of action of the resultant.

10165. (W. J. GREENSTREET, M.A.)—Find the locus of the vertices of the triangles circumscribing a fixed circle, and of which the orthocentre is a fixed point.

Solution by the PROPOSER; Professor MATZ; *and others.*

Take α, β, γ the angular parameters of the points of contact; then the orthocentre is given by

$$4x_1\cos\tfrac{1}{2}(\beta-\gamma)\cos\tfrac{1}{2}(\gamma-\alpha)\cos\tfrac{1}{2}(\alpha-\beta)$$
$$= 4R\cos\alpha\cos\beta\cos\gamma + R\left[\cos\alpha+\cos\beta+\cos\gamma-\cos(\alpha+\beta+\gamma)\right],$$
$$4y_1\cos\tfrac{1}{2}(\beta-\gamma)\cos\tfrac{1}{2}(\gamma-\alpha)$$
$$= 4R\sin\alpha\sin\beta\sin\gamma + R\left[\sin\alpha+\sin\beta+\sin\gamma+\sin(\alpha+\beta+\gamma)\right]$$

If the orthocentre be d, 0 and if (x, y) be the pole of the chord $\beta\gamma$, we get from the above equations

$$y \sin \alpha (x-d) + \cos \alpha (2R^2-y^2-dx) + R(x-d) = 0,$$
$$\sin \alpha (2R^2-x^2) + xy \cos \alpha + Ry = 0.$$

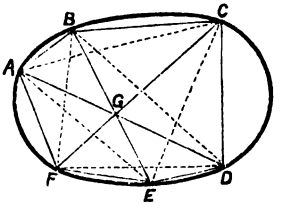
Eliminating α, we get, for the locus of the vertex opposite $\beta\gamma$,

$x^2(R^2-d^2) + R^2y^2 + 2dR^2x = R^2(4R^2-d^2)$—a symmetrical expression, therefore the locus of all three vertices.

10650. (R. KNOWLES, B.A.)—If the three diagonals of a hexagon, inscribed in a conic, pass through the some point G; prove that the points of intersection of its opposite sides are on the polar of G.

Solution by Professors ANDERSON, ZERR, *and others.*

Let ABCDEFA be a hexagon inscribed in a conic, and let AD, BE, CF meet in G. Then the quadrilateral ABDE is inscribed in the conic, and therefore the intersection of AB and DE lies on the polar of G. In like manner, the points of intersection of BC, EF, and of AF, DC lie on the same polar.



10806. (Professor WOLSTENHOLME, Sc.D.)—If x, y, z be three finite quantities, such that $x+y+z=0$.....(1),

$(ax+by+cz)^2 = 4(bc+ca+ab)(yz+zx+xy)$, $ax^2+by^2+cz^2=0$...(2, 3),

prove that $(a+b+c)(bc+ca+ab) = 9abc$.

Solution by Professors SARKAR, AIYAR; *and others.*

From (1), $2xy = z^2-x^2-y^2$, &c. = &c. Substituting these values of xy, &c. in (2), collecting coefficients of x^2, y^2, z^2, and putting $a+b+c=\lambda$, we have $(a\lambda+3bc)x^2 + (b\lambda+3ca)y^2 + (c\lambda+3ab)z^2 = 0$;

hence, from (3), $bcx^2+cay^2+abz^2=0$.....(4).

From (3), (4), we have $\frac{x^2}{a(b^2-c^2)} = \frac{y^2}{b(c^2-a^2)} = \frac{z^2}{c(a^2-b^2)}$.....(5);

from (5), (1), $\{a(b^2-c^2)\}^{\frac{1}{2}} + \{b(c^2-a^2)\}^{\frac{1}{2}} + \{c(a^2-b^2)\}^{\frac{1}{2}} = 0$.

Transposing last term, squaring, &c., we obtain

$$2\{ab(b^2-c^2)(c^2-a^2)\}^{\frac{1}{2}} = (c+a)(c+b)(a-b);$$

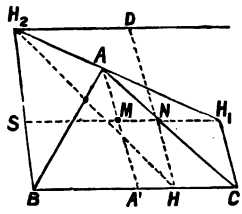
whence $$a^2(b+c)+b^2(c+a)+c^2(a+b) = 6abc,$$

and, adding $3abc$, the result follows.

9306. (Professor Ignacio Beyens.)—Si dans un triangle quelconque ABC, AA′ est la médiane correspondante au côté BC, et l'on mène à AA′ une parallèle qui coupe BC en H, AC en N, BA en D; démontrer que les points symétriques de H par rapport aux milieux de NC, BD sont deux points H_1, H_2 symétriques par rapport au sommet A du triangle proposé.

Solution by the Proposer.

La solution est indiquée par les proportions suivantes et la figure ci-jointe:



$$\frac{NH}{AA'} = \frac{CH}{CA'}, \quad \frac{DH}{AA'} = \frac{BH}{BA'},$$

$$SH_2 = H_2B - SB = DH - NH$$

$$= \frac{AA'}{BA'}(BH - CH),$$

ou $$SH_2 = \frac{AA'}{BA'} \cdot 2A'H \ldots\ldots\ldots (1).$$

$$AM = AA' - A'M = AA' - NH$$

$$= AA' - \frac{AA'}{CA'} \cdot CH = AA' \frac{(CA' - CH)}{CA'} = AA' \cdot \frac{A'H}{CA'} \ldots\ldots\ldots (2);$$

et divisant (1), (2) on a $\frac{SH}{AM} = 2$. Mais $SH_1 = BC$, $MH_1 = A'C$; donc $\frac{SH_2}{AM} = \frac{SH_1}{MH_1} = 2$; donc les trois points H_1, A, H_2 sont dans la même droite, et $AH_1 = AH_2$.

10455. (Professor Déprez.)—Les milieux des côtés d'un triangel ABC, le centre I du cercle inscrit, le centre O du cercle circonscrit, et le centre de gravité du périmètre de ABC sont six points d'une même hyperbole équilatère.

Solution by Prof. Anderson, M.A.; Belle Easton, B.Sc.; *and others.*

Every equilateral hyperbola through the middle points D, E, F of the sides passes through the orthocentre of DEF, which is the centre of the circumcircle of ABC.

Taking, now, DEF as triangle of reference, the equation of the hyperbola may be written $\frac{l}{\alpha}+\frac{m}{\beta}+\frac{n}{\gamma}=0$, with the condition

$$l\cos A+m\cos B+n\cos C=0.$$

We can now make the hyperbola pass through the centre of gravity of the perimeter ABC, which is the centre of the incircle of DEF, by imposing the condition $l+m+n=0$.

Hence the equation of the hyperbola is

$$\frac{\cos B-\cos C}{\alpha}+\frac{\cos C-\cos A}{\beta}+\frac{\cos A-\cos B}{\gamma}=0,$$

or

$$\frac{s-a}{\alpha}+\frac{s-b}{\beta}+\frac{s-c}{\gamma}=0.$$

But the coordinates of the incircle of ABC are easily found to be proportional to $s-a$, $s-b$, $s-c$.

10584. (E. M. Langley, M.A.)—O is any point in the plane of a triangle ABC. If AO, BO, CO cut the circumcircle again in A′, B′, C′, and D, E, F are the projections of O on BC, CA, AB, show (1) that the triangles A′B′C′, DEF are directly similar, and that O in A′B′C′ corresponds with the isogonally inverse point of O in DEF; also, (2) use the theorem to demonstrate the existence of cosymmedian triangles having the sides of each proportional to the medians of the other.

Solution by H. J. Woodall; E. M. Langley; *and others.*

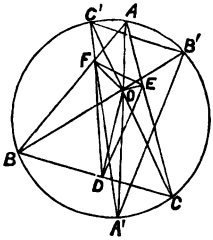
1. $B'A'C'=B'A'O+OA'C'=B'A'A+AA'C'$
$=B'BA+ACC'$,
$EDF=EDO+ODF$;

but EODC is cyclic; therefore

$$EDO=ECO=ACC',$$

and $EDF=ACC'+B'BA=B'A'C'$.

Therefore A′B′C′, DEF are directly similar, and O in A′B′C′ corresponds to the isogonally inverse point of O in DEF.



[Otherwise:— $\angle OA'C'=ACC'=EDO$, because EODC is cyclic; and similarly $\angle B'A'O=ODF$; therefore $\angle B'A'C'=EDF$.

2. If O is symmedian point of ABC, it is known to be the centroid of DEF; hence it is the symmedian point of A′B′C′. Also OD, OE, OF, which are proportional to the medians of DEF, and therefore of A′B′C′, are known to be proportional also to the sides of ABC.]

APPENDIX I.

ELEMENTARY GEOMETRICAL NOTE ON THE "FRÉGIER-POINT" OF AN ELLIPSE.

By R. F. Davis, M.A.

1. Let PNP′ be a double ordinate to an ellipse (fig i.). Then the real common chords of the ellipse and a point-circle at P will be the tangent at P to the ellipse and a straight line RR′ (exterior to the ellipse) parallel to the tangent at P′, these two straight lines being of course equally inclined to the major axis.

2. If ω,ω' are the two imaginary points of intersection of the ellipse and point-circle each lying on RR′; then, since RR′ is parallel to the tangent at P′, it follows that CP′, if produced, will bisect $\omega\omega'$ (for the locus of the middle points of parallel chords of an ellipse is a fixed diameter). Since then U is the mid-point of $\omega\omega'$, it follows (from the properties of the circle) that the perpendicular to $\omega\omega'$ (or RR′) at its mid-point U passes through P, the centre of the point-circle. Hence PU is perpendicular to RR′, or P′T.

Thus, if we are given P, we can determine U, and thence RR′.

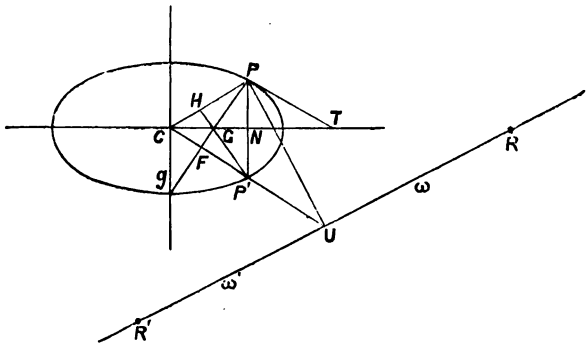


Fig. I.

3. From any point O in RR′ (fig. ii.) draw the tangents OQ, OQ′ meeting the tangent at P in T, T′. Then by a well-known property of the ellipse $TP^2 : TQ^2$ is as the squares of diameters parallel to TPT′, OQT

..... $O\omega\omega'$, OQT

$= O\omega.O\omega' : OQ^2$,

since TPT', $O\omega\omega'$ being equally inclined to the axis are parallel to equal diameters. But $O\omega . O\omega'$ = (tangent)^2 from O to the point-circle
$= OP^2$
Hence $TP^2 : TQ^2 = OP^2 : OQ^2$,
or the angle OPT is bisected by PQ.

4. Similarly the angle OPT' is bisected by PQ'; hence QPQ' is a right angle. Thus, if from any point O on the line RR' tangents OQ, OQ' be drawn to the ellipse, the chord of contact QQ' always subtends a right angle at P.

5. By polar properties, QQ' passes through a fixed point F (the polar of RR') lying within the ellipse upon the diameter CP'. This point is generally known by the name of its discoverer, Frégier. The line RR' might very appropriately be called the Frégier-line; so that the Frégier-point and line are related as pole and polar.

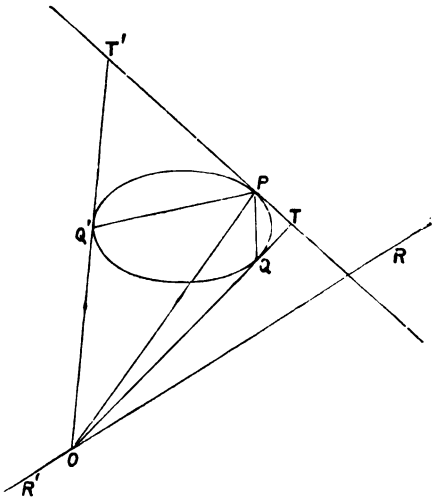


FIG. II.

6. The point F lies on the normal at P. For when Q moves up to P, PQ becomes the tangent at P, and PQ' (which is perpendicular to PQ) the normal chord at P. Hence the normal chord at P is itself one position of QQ'.

7. Or thus: Since (fig. i.) PU is perpendicular to RR', it is parallel to the normal P'G.
Hence CU : CP' = CP : CH (if P'G meets CP in H)
= CP' : CF
$\therefore$ CF . CU = CP'^2.

8. PF is evidently the harmonic mean between PG and Pg.
Now PG = (b / a) . CD and Pg = (a / b) . CD,

hence $PG : PF : Pg = b^2(a^2 + b^2) : 2a^2b^2 : a^2(a^2 + b^2)$

and $PG : GF = b^2(a^2 + b^2) : b^2(a^2 - b^2)$

$= (a^2 + b^2) : (a^2 - b^2)$.

But CG, Cg are the bisectors of the angle PCF;

$\therefore$ $CP : CF = (a^2 + b^2) : (a^2 - b^2)$,

or $CF = \frac{a^2 - b^2}{a^2 + b^2} . CP$

$= \ldots\ldots\ldots CP'$.

Hence, as P moves positively round the four quadrants, the Frégier-point F describes a similar, similarly situated, and concentric ellipse of $(a^2 - b^2) / (a^2 + b^2)$ its linear dimensions with equal but opposite angular velocity. Similarly for the envelope of the Frégier-line.

9. The polar reciprocal of the ellipse with respect to the point P is a parabola whose focus is the reciprocal of the line RR', and whose directrix is the reciprocal of the point F. In fact the theorem in (4) is the reciprocal of the property of the parabola, that if a chord be drawn through the focus, the tangents at its extremities are at right angles and intersect on the directrix.

10. From the demonstration in (3) it may be noticed that if pp' be any chord bisected by CP, and upon pp' produced a point O be taken such that $Op . Op' = OP^2$, then the locus of O is the Frégier-line corresponding to P.

APPENDIX II.

UNSOLVED QUESTIONS.

4482. (Professor Wolstenholme, Sc.D.)—Prove that (1) the equation of a cardioid may be written

$$(x+iy)^{-\frac{1}{2}}+(x-iy)^{-\frac{1}{2}}+a^{-\frac{1}{2}}=0, \text{ or } X^{-\frac{1}{2}}+Y^{-\frac{1}{2}}+Z^{-\frac{1}{2}}=0,$$

where $i=\sqrt{(-1)}$; and (2) that of the tricusped hypocycloid

$$(x+y\sqrt{3}+a)^{-\frac{1}{2}}+(x-y\sqrt{3}+a)^{-\frac{1}{2}}+(a-2x)^{-\frac{1}{2}}=0,$$

whence we see that either is the projection of the other.

4483. (Professor Crofton, F.R.S.)—1. Let 1, 2, 3, 4 be points on a circle; through any internal point P draw the circles 1P3, 2P4; if circles be now inscribed in each of the four curvilinear triangles into which the circle (1234) is divided, show that their points of contact with (1234) are the vertices of the two bicircular quartic ovals, foci 1, 2, 3, 4, which meet (orthogonally) in P.

2. The above two bicircular ovals may be thus drawn:—Draw any circle touching the two arcs P1, P2 (for instance); find the pair of points which are *co-inverse** to this circle and (1234); both of these are points on the oval which bisects the angle 1P2.

3. If a vertex, A, of a bicircular oval be given, it may be drawn thus: Supposing A to lie on the arc 12, draw *any* circle touching (1234) at A; draw two circles to touch the former, one through 14, the other through 23; they meet each other in two points on the oval.

These methods apply to Cartesians and also to conics.

4489. (The late T. T. Wilkinson, F.R.A.S.)—P is a given point in the side BC of the triangle ABC. From P draw lines including a given angle and meeting AB, AC in D, E, so that $\triangle DBP : \triangle EPC = m : n$.

4490. (R. Tucker, M.A.)—A circular hoop is bisected and has its extremities connected by a stretched string without weight. The system, which is capable of motion in a vertical plane, is in equilibrium, and a small animal placed at the centre of the string crawls towards the rim: find the locus of the centroid of the new system.

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* [That is, inverse to one another with regard to both circles: they may be found in various ways, given the two circles.]

4494. (N'Importe.)—Two force-points endowed with inertia, and subject to no forces but their mutual attraction and repulsion, are approaching each other at the interval r with the common velocity v, under the influence of the accelerating force $mm_1(r-a)f_n(r) : r^{n+3}$, where $f_n(r)$ is a rational and integral function of r of the nth degree, whose roots are all positive, and each less than a. Required the motion.

4495. (Professor Minchin, M.A.)—Prove that the envelope of the diameter of the osculating conic of a semi-cubical parabola is another semi-cubical parabola.

4496. (The late T. Cotterill, M.A.)—A and S are fixed points, P a variable point on a given line d; show that the locus of the intersection of AP with the line of reflexion of AS to SP is the conic of which A is a point, S a focus, and d the corresponding directrix.

4509. (Professor Crofton, F.R.S.)—If $l\rho + m\sigma + n\tau = 0$ represent the serpentine branch of a circular cubic, show that $l\rho^2 + m\sigma^2 + n\tau^2 = 0$ represents the tangent to the curve at the centre of the four concyclic foci.

4510. (The late C. W. Merrifield, F.R.S.)—If the shortest distances between the opposite edges of a tetrahedron meet in a point, then, in general, the rectangles between their segments are equal. But if two of them (only) be orthogonal to the third, then those two are bisected and equal to one another; and further, if all three be orthogonal, they bisect one another; and also the edges of the tetrahedron, the four corners of which are the alternate corners of a rectangular parellelepiped.

4511. (Rev. T. P. Kirkman, M.A., F.R.S.)—Of the 68 janal anaxive 16-acral 12-edra, one is

ABCDE *a*HGF*b* *b*FEA*c* *a*CH GFED GDCH
abcde A*hgf*B B*fea*C A*ch* *gfed* *gdch*

which has three different 5-gons, two different 4-gons, and one triangle, each face, summit, and edge being diametrically opposite to its reflected image, the solid having no zone nor pole. There are n (>5 and <9) other janal anaxive 16-acral 12-edra, each having three different 5-gons, two different 4-gons, and one triangle. Required the twelve faces of each of the n solids, described as above in pairs by their consecutive summits, with the same sixteen letters.

4514. (Rev. W. Roberts, M.A.)—Let F_1, F_2, F_3 be the foci of a system of confocal Cartesians. Prove that if $\frac{F_1F_2}{F_2F_3} = \frac{2-\sqrt{3}}{2+\sqrt{3}}$, the system will be cut under an angle of 60° by a system of algebraic curves.

4571. (Professor Crofton, F.R.S.)—A line of given length is to be drawn across a given angle, and passing through a given point: a line drawn by guess through the point is found to differ from the required length by a given small quantity. Give an easy rule for obtaining (graphically) from this first trial, a very near approximation to the true solution.

4573. (J. J. Walker, F.R.S.)—Referring to Question 4530, show that the segments of the lines drawn from the point in which the tangent to the circumscribing circle at A meets the base BC of the triangle ABC, parallel to the internal and external bisectors of that angle, are proportional to the roots of the equations

$$2x^2\cos\tfrac{1}{2}A - a(b-c)x + 2b^2c^2 = 0, \text{ and } 2x^2\sin\tfrac{1}{2}A - a(b+c)x + 2b^2c^2 = 0,$$

respectively, the discriminants of which are

$$(b+c)^2(4bc-a^2), \text{ and } -(b-c)^2(4bc+a^2), \text{ respectively.}$$

4585. (Rev. T. P. Kirkman, F.R.S.)—If n be an odd prime, prove that the equation $x_1^n = x_2^n + x_3^n + x_4^n + \ldots\ldots + x_n^n$ is impossible in integers, unless $n-2$ of the terms of the right member are zeros. The famous and unproved negative of Fermat is a case of this.

4597. (Professor Burnside, M.A.)—Show that the degree of the locus of the foci of a system of conics, subject to any four conditions, is three times as great as that of the locus of their centres.

4608. (Rev. J. R. Wilson, M.A.)—A heavy elastic string PAQ of weight W, modulus λ, and natural length a, is placed over the free end of a smooth thin bar OA ($=a$), movable about its lower extremity O. A portion of the string (PA) rests upon the bar; the remainder (QA) hangs freely. The bar is now moved round O, and the string allowed to take up its positions of rest. If O be taken as origin, and the axis of x horizontal, show that the locus of Q is

$$y\left\{2a^2 + 2a(a^2-x^2)^{\frac{1}{2}} - x^2\right\} = (a^2-x^2)\left\{a - \tfrac{1}{2}\lambda^{-1}a\mathrm{W} + (a^2-x^2)^{\frac{1}{2}}\right\}.$$

4609. (Prof. Christine Ladd, B.Sc.)—A circle of radius R is completely surrounded by n circles. These are in turn surrounded by successive rows of circles, each circle touching the two adjacent circles of its own row, and also two adjacent circles of the next inner row. Required the radius of the circles of the mth row.

4611. (J. M. Johnson, B.A.)—A circular hoop rolls along a road rough enough to prevent sliding, and is prevented from changing its form under impressed forces, as for example, a thin tire tightly stretched on a rigid wheel; find the tangential and normal forces, and the breaking moment due to rotation only at any point of the hoop. Show that, if T, n, g denote their values at a point of the hoop distant θ from the highest point, r being the radius of the hoop, v the velocity of its centre, and E the mass of unit length of the hoop,

$$\mathrm{T} = mv^2(1 + 2\cos\theta), \quad n = 2mv^2\sin\theta, \quad g = 3mv^2r\cos\theta.$$

Calculate the greatest numerical value these have in tons per square inch, when the diameter of the hoop is 3 ft. 6 in. and its velocity one of 40 miles an hour. Taking the specific gravity of the tire as not very different from 7 compared with water, we have $\mathrm{T} = \frac{3}{4}$ ton per sq. in. at top, $n = \frac{4}{5}$ ton per sq. in. at bottom, g = about 14 tons at top and bottom.

4625. (The late Rev. J. Blissard.)—Prove that

(1) ..... $\left\{\sin x \cdot \frac{\Delta}{1} - \frac{\sin 2x}{2\cos x} \cdot \frac{\Delta^2}{2} + \frac{\sin 3x}{(2\cos x)^2} \cdot \frac{\Delta^3}{3} - \&c.\right\} 0^n = 0$ (n even),

(2) ..... $\left\{\cos x \cdot \frac{\Delta}{1} - \frac{\cos 2x}{2\cos x} \cdot \frac{\Delta^2}{2} + \frac{\cos 3x}{(2\cos x)^2} \cdot \frac{\Delta^3}{3} - \&c.\right\} 0^n = 0$ (n odd and >1).

4634. (A. Martin, LL.D.)—A circle of radius r has internal contact with a circle of radius R; and a triangle is drawn having its vertices in the circumference of the outer circle, and one side tangent to the inner circle; find the average area of the triangle.

4635. (Professor Hudson, M.A.)—Show (1) how to find the envelope of the pedals of a given curve when the given point moves along another curve, and prove that it is the envelope of the pedals of the latter curve when the given point moves along the former; and (2) find this envelope if the curves are $y^2 = 4ax$, $x^2 + y^2 = a^2$.

4644. (The late Dr. Booth.)—A quadrant of a great circle moves between two great circles inclined to each other at the angle $\frac{1}{2}\pi + \theta$. Find its envelope.

4646. (Rev. W. Roberts, M.A.)—Denoting by ρ^2, μ^2, ν^2 the roots of the equation in t, $\frac{x^2}{t} + \frac{y^2}{t-b^2} + \frac{z^2}{t-c^2} = 1$, prove that the equations $\rho^2 - \mu^2$ = const., $\rho^2 - \nu^2$ = const. represent the sheets of a wave surface.

4672. (Rev. W. Roberts, M.A.)—Denoting, as usual, the elliptic coordinates in two dimensions by μ and ν, which are respectively half the sum, and half the difference, of the distance of a point from two fixed points, the distance between which is $2b$; prove that the equation

$$\int \left(\frac{\mu^2 - a^2}{\mu^2 - b^2}\right)^{\frac{1}{2}} d\mu \pm \int \left(\frac{a^2 - \nu^2}{b^2 - \nu^2}\right)^{\frac{1}{2}} d\nu = c + (\mu^2 + \nu^2 - b^2)^{\frac{1}{2}}$$

represents the first negative pedal, with reference to the centre, of the involute of the ellipse, a and c being constant quantities.

4688. (Professor Genese, M.A.)—O is a fixed point in the plane of a circle, P any point on the circumference, Q is taken on OP so that $OQ = \frac{a^2}{OP} + \lambda$ OP. (a, γ, λ being constants); prove that, if QN be drawn perpendicular to OQ, its envelope will be a conic. Ex. 1: QN the polar of P with respect to a fixed circle. Ex. 2: QN the radical axis of a fixed circle and a circle with P as centre and constant radius.

4703. (Rev. W. Roberts, M.A.)—Prove that the equation of the second negative pedal of an ellipse, with reference to the focus as origin, is found by equating to zero the discriminant of the following cubic in θ:—

$$(2a + 2c - \theta)\ (2a - 2c - \theta)\ \theta + c\theta x + (a\theta - b^2)\ (x^2 + y^2)^{\frac{1}{2}} = 0,$$

a, b being the semi-axes, and c the distance of the focus from the centre.

4704. (The late T. Cotterill, M.A.)—If the perpendicular y' from a point P′ on the line y is equal to the perpendicular x from a point P on the line x, and the line PP′ passes through the intersection of x and y, then there is a quartic correspondence between the points P, P′. Find its principal points and principal lines. Explain the connexion of the curves $x^{n+1} = ay^n$ (n an integer).

4705. (The late Rev. J. Blissard.)—Prove that

$$\pi = \frac{94}{30} + 1 \cdot 2 \Sigma \frac{(-1)^n}{4^{n-1}} \cdot \frac{20n^2 + 33n + 13}{(4n+1) \ldots\ldots (4n+5)}$$

$$\text{or} = \frac{22}{7} + 1 \cdot 2 \cdot 3 \cdot 4 \Sigma \frac{(-1)^{n+1}}{4^{n-1}} \cdot \frac{20n^2 + 45n + 25}{(4n+1)(4n+2) \ldots\ldots (4n+7)}$$

$$\text{or} = \frac{1979}{630} + 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot \Sigma \frac{(-1)^n}{4^{n-1}} \cdot \frac{20n^2 + 71n + 41}{(4n+1)(4n+2) \ldots\ldots (4n+9)},$$

and so on; the integration being taken from $n = 1$ to $n = \infty$.

4709. (J. Griffiths, M.A.)—Trace the curve whose tangential equation is

$$(\sin A \cdot \mu\nu + \sin B \cdot \nu\lambda + \sin C \cdot \lambda\mu)^2 = 4\lambda\mu\nu (\lambda \cos A + \mu \cos B + \nu \cos C).$$

4713. (S. Roberts, M.A.)—Show that the locus of centres of curves of the nth degree, which pass through the points of intersection of two curves of the same degree, is of the degree $2(n-1)$. [According to M. Chasles, the centre is the centre of mean distances of the points of contact of parallel tangents.]

4728. (R. A. Roberts, M.A.)—Prove that the anharmonic function of the tangents from any point to the cubic $(x+y+z)^3 - 27xyz = 0$, is equal to the anharmonic function of the cubic of the system $(x+y+z)^3 + \lambda xyz = 0$, which passes through the point.

4732. (R. Tucker, M.A.)—Three straight lines pass through a point O; a circle also passing through O turns about this point; the diametral feet-perpendicular lines of one point of section with respect to the other two lines intersect on an ellipse having O for its centre. Find also the areas of the several ellipses.

4745. (C. R. Rippin, M.A.)—Three equal balls A, B, C, are placed on a smooth table, so that the centres of A and B are in a straight line which touches the circumference of C; and B is also in contact with a smooth vertical plane to which AB is normal. A fourth equal ball D is made to impinge upon A with velocity u in the line AB. Find the initial motion of each ball; the elasticity of the balls and plane being the same (coefficient e.) Also show that, whatever be the value of e, the angle between the initial motions of A and C $= \tan^{-1}(\frac{1}{7}\sqrt{3})$.

4746. (Professor Hudson, M.A.)—A ray of light traverses a medium in which the density at any point is a function of (r, θ), the polar coordinates of the point; prove that, if μ be the refractive index,

$$\frac{\mu}{\rho} = \frac{\cos\psi}{r} \frac{d\mu}{d\theta} - \sin\psi \frac{d\mu}{dr},$$

where ρ is the radius of curvature of the path of the ray, and ψ the inclination of its tangent to the radius vector.

4754. (The late Professor Clifford, F.R.S.)—1. The distances (r, s, t) of a variable point on *one* focal curve of an anallagmatic quartic surface from any three fixed points on another focal curve of the same surface, are connected by a relation of the form $lr + ms + nt = 0$ (*i.e.*, a relation of the same form as that which connects the distances of a variable point on a circle from three fixed points on the circle).

2. The distances of a variable point on an anallagmatic quartic surface

from four fixed foci of the surface are connected by a relation of the same form as that which connects the distances of a variable point on a sphere from four fixed points on a sphere.

3. The four points in which an anallagmatic quartic curve is cut by any circle may be taken as the foci of an anallagmatic quartic curve which passes through any four concyclic foci of the original curve. (This is true on a plane or on a sphere.)

4. The curve in which an anallagmatic quartic surface is cut by any sphere may be taken as the focal curve of an anallagmatic quartic surface which passes through any one focal curve of the original surface.

5. It is required to find the property of anallagmatic quartic curves which corresponds to the property of conics, $SP \cdot HP = CD^2$.

6. If two anallagmatic quartic curves or surfaces A and B are such that a confocal to A can be inscribed or sub-inscribed to B: then also a confocal to B can be inscribed or sub-incribed to A.

DEF.—One anallagmatic curve is *inscribed* to another when it touches it in four points on a circle. One surface is inscribed to another when it touches it all along its curve of intersection with a sphere. One surface is *sub-inscribed* to another when it touches it in four points on a circle.

4766. (The late T. Cotterill, M.A.)—1. The reflexions of a point on a circle to the three pairs of lines joining four points on the same circle are the opposite points of a quadrilinear (system of four lines in a plane), and its reflexions to these four lines are collinear.

2. The three pairs of lines joining the point to the opposite points, and four pairs formed by joining the point to each of the four co-circular points and its corresponding point on the line, together with the diameter of the circle through the point and its perpendicular on the line, form 8 pairs of distances with equal products and the same bisectors, which are parallel to the bisectors of the pairs of lines through the four points on the circle.

4773. (Professor Crofton, F.R.S.)—Given any two convex boundaries of lengths L and L′, the latter inside the former; if Ω be the area of the inner, and if we denote the function $x - \sin x$ by u_x, prove that

$$\iint (u_{\theta+\phi} + u^{\theta+\psi} - u_\phi - u_\psi) d\mathrm{S} + 2\iint \theta \, d\mathrm{S} = \mathrm{LL}' - 2\pi\Omega,$$

where θ is the angle between two tangents drawn to the curve L′ from any point P, and ϕ, ψ the angles between the same tangents and two other tangents drawn from P to the curve L; the first integral extending over the whole space outside L, and the second over the space between L and L′.

4777. (The late T. Cotterill, M.A.)—Show that there is a complete analogue on the sphere to Newton's organic description of a conic in a plane. On a sphere let the points P, P′ subtend an angle A at the point A, and an angle B at the point B, C being the position of P′ when P is on the arc AB; then, if x, y, z, x', y', z' be the sines of the perpendiculars from P, P on the sides of the spherical triangle, ABC, prove that

$$-z'z = x' (x + 2\cos \mathrm{BZ}) = y' (y + 2\cos \mathrm{AZ}).$$

What is the meaning of these equations when the variables coincide?

4783. (Professor Genese, M.A.)—A circle passes through a fixed point, and a point of its circumference describes a straight line. Prove

that the locus of any other point of the circumference is in general a pedal of a conic; but if the straight line pass through the fixed point, the locus will be a straight line. Find also the locus of any point carried by the circle, and the envelope of the line joining the points.

4786. (A. Martin, LL.D.)—Find the mean distance of a given point in the surface of a circle (1) from all points in its circumference, and (2) from all other points in its surface.

4808. (The late T. Cotterill, M.A.)—Let aa', bb', cc' be fixed points, and conics through aa', cc', and bb', cc', intersect again in p and p'; then, if p describe a right line, p' will describe a curve of the 6th orner, having triple points at cc', double points at aa', bb', and a single point at the intersection of aa' and bb'. The locus of the points of contact of two such conics which touch is a curve of the 4th order having nodes at cc', and passing through aa', bb'; the envelope of the tangents at these points is a class quartic having the lines through aa' and bb' for double tangents. What do these curves become, if the six points are on the same conic?

4814. (Professor Burnside, M.A.)—Prove that all the invariants of a system of three conics are expressible in terms of Θ_{123}, and similar quantities derived by altering the suffixes in every way; where

$$\Theta_{123} \equiv a_1 (b_2c_3 + b_3c_2 - 2f_2f_3) + b_1 (c_2a_3 + c_3a_2 - 2g_2g_3) + c_1 (a_2b_3 + a_3b_2 - 2h_2h_3) + 2f_1 (g_2h_3g_3h_2a_2f_3a_3f_2) + 2g_1 (h_1f_2h_2f_3b_2g_3b_3g_2) + 2h_1 (f_2g_3f_3g_2c_2h_3c_3h_2),$$

and the equations of the conics are written in the form

$$(a_1b_1c_1f_1g_1h_1)\ (xyz)^2 = 0, \text{ \&c.}$$

4823. (Sir R. Ball.)—Prove that the centre of gravity of a rigid body, which is performing small oscillations about a fixed point under the action of gravity, has a movement compounded of two vibrations at right angles to each other.

4827. (S. Roberts, M.A.)—Representing by R (x, y, z) the reciprocal of u, explain (1) the nature of the locus $\mathrm{R}\left(\frac{du}{dx}, \frac{du}{dy}, \frac{du}{dz}\right) = 0$, and account for its degree; and (2) give corresponding explanations relative to space of three dimensions.

4850. (J. C. Glashan.)—Trace the curve $r = \epsilon^{\pm(\frac{1}{2}\pi^2 - \theta^2)^{\frac{1}{2}}}$.

[Mr. Glashan wishes to know whether the properties of the curves

$$r = \epsilon^{\pm(a^2 - \theta^2)^{\frac{1}{2}}}, \quad r = \epsilon^{\pm(\theta^2 - a^2)^{\frac{1}{2}}}$$

have been investigated in cases other than $a = 0$; and if so, by whom and in what work, and by what name the curves are known. He finds them of great importance in the application of Double Algebra to Geometry.]

4858. (Rev. A. F. Torry, M.A.)—A rod is placed with one extremity at the middle point of the line joining two centres of force, which attract inversely as the square of the distance, the rod being at right angles to this line: find the velocity with which the centre of the rod will cross this line. If the rod were placed at right angles to the line joining the centres, and very near the position of equilibrium; determine the time of its oscillation about that position.

4866. (R. A. Roberts, M.A.)—If from any point (x', y', z') tangents be drawn to the curve $x^2y^2+y^2z^2+z^2x^2=0$; prove that their six points of contact lie on the conic $(y'^2+z'^2)x^2+(z'^2+x'^2)y^2+(x'^2+y'^2)z^2+x'y'xy+y'z'yz+z'x'zx=0$.

4878. (Sir R. Ball.)—If a rigid body having three degrees of freedom be in equilibrium under the action of gravity, the restraints must be such as would permit the body to be rotated about an axis passing through the centre of gravity.

4895. (M. Jenkins, M.A.)—In Degen's table of the quotients to be used to form the convergents to the value of $N^{\frac{1}{2}}$, where N is any non-square integral number from 1 to 1000, it is seen that the number of quotients in the period (excluding the first quotient which does not recur) never exceeds $2a$ when N lies between a^2 and $(a+1)^2$. Can this be proved generally?

4905. (R. A. Roberts, M.A.)—If from any point of the cubic

$$(a-bc)\,x(y^2-z^2)+(b-ca)\,y(z^2-x^2)+(c-ab)\,z(x^2-y^2)=0$$

tangents be drawn to the quartic

$$x^2y^2+y^2z^2+z^2x^2+2xyz(ax+by+zc)=0,$$

prove that their six points of contact lie on a conic.

4907. (Professor Cavallin.)—If in the plane of a *convex* quadrilateral we proceed to find a point such that the sum of its distances from the angular points may be a minimum, we see at once that the point sought is the point of intersection of the diagonals, but this does not hold good for a *concave* quadrilateral. Find the solution in this case.

4908. (J. J. Walker, F.R.S.)—A lever AB, whose fulcrum C coincides with its centre of weight, is in equilibrium under the action of forces from A and B to fixed points D, E. Show that the equilibrium will be stable if D, E are within semicircles described on AC, BC respectively. (Practically, such an equilibrium may be established by cords carried from A and B round fixed pulleys at D, E; weights being attached to the free ends of the cords.)

4918. (The late T. Cotterill, M.A.)—A point P is the centre of a sphere cutting orthogonally a fixed sphere, and determining by its intersections with another fixed sphere a radical plane Π. Show that such points and planes are correlative; and having given a plane Π, determine the corresponding point P.

4920. (Alpha.)—Si l'on désigne par X_p^n le nombre total de manières dont un polygone convexe de n côtés peut être décomposé en p parties au moyen de $(p-1)$ diagonales qui ne se coupent pas dans l'intérieur du polygone, on a

$$X_p^n=\frac{1}{p}\cdot\frac{n(n+1(n+2)\ldots(n+p-2)}{1\cdot2\cdot3\ldots(p-1)}\cdot\frac{(n-3)(n-4)\ldots(n-p-1)}{1\cdot2\cdot3\ldots(p-1)}.$$

Pour $n=p-2$ on retrouve la formule d'Euler démontrée dans le Journal de Liouville, tom. III. (1838), p. 505 et p. 547.

4921. (R. Tucker, M.A.)—Find curves whose orthogonal radii of curvature are such that (1) sums are constant, (2) rectangles are constant.

4941. (R. A. Roberts, M.A.)—If A, B, C, D are the areas of the faces of a tetrahedron formed by a triangle circumscribed to a parabola, and a point on the focal parabola, prove that $A + C = B + D$.

4951. (Sir R. S. Ball, F.R.S.)—A rigid body acted upon by gravity rotates around a fixed point. Show that, when performing small oscillations, its movements are compounded of vibrations around two normal axes. Show that these normal axes are determined by the following construction. Draw the momental ellipsoid A; and draw another ellipsoid B, the axes of which are proportional to the squares of those of A, and in the same direction. Draw an ellipsoid A, a plane conjugate to the vertical direction. This plane cuts A and B in two ellipses, the common conjugate diameters of which are the normal axes.

4957. (Professor Hudson, M.A.)—A mass M of elastic fluid in a vertical circular cylinder of radius r is confined by a piston of given weight; the temperature is maintained constantly less than the temperature at the base by a quantity which varies as the height k above the base; determine the pressure at any point, and prove that the whole pressure in the curved surface of the cylinder is $\frac{2Mk}{r}$.

4960. (The late T. Cotterill, M.A.)—In a plane, abc and $a'b'c'$ are two triangles in which ab and $a'b'$ meet in d. A variable line cuts ca, cb, $c'a'$, $c'b'$ in α, β, α', β' respectively. Show that the conics $(dab'\ a\beta')$ and $(da'b\ a'\beta)$ intersect in points on the cubic locus of points at which the three point-pairs aa', bb', cc' subtend angles in involution, and that the lines connecting the three variable points of intersection of the conics cut the cubic again in three collinear points.

4961. (R. Tucker, M.A.)—A circle (O) through the foci of a rectangular hyperbola cuts the curve in P, Q on one branch, and P′, Q′ on the other branch, and the asymptotes in K, L, K′, L′. If QQ′ cut CO in M and PN be the ordinate of P, prove that (1) MN = radius of (O); (2) KL touches the hyperbola; (3) the product of the radii of curvature of the hyperbola at P, Q varies as the cube of the radius (r) of (O); and (4) $CP^2 + CQ^2 = 2r^2$.

4964. (Rev. W. A. Whitworth, M.A.)—A regular "screw-polygon" is formed of equal straight lines. Any two adjacent sides contain an angle $\pi - \theta$, and the planes containing any side and either of the adjacent sides contain an angle ψ. Show that, if there be n sides in a convolution,

$$\cos\frac{\pi}{n} = \cos\frac{\theta}{2}\cos\frac{\psi}{2}.$$

4967. (Rev. W. Roberts, M.A.)—Given a system of confocal spherical conics, in which the distance between the foci is $\frac{1}{2}\pi$; prove that the family of curves cutting them at an angle of 45° is algebraic, and determine its equation.

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